Introduction to Mathematical Biology
Exercises 1.1-1.5

1.1 *Exponential decay.* A typical protein of a yeast cell decays at the constant rate $\mu = 1/\text{hour}$. Calculate the probability that (i) a protein molecule decays within one hour; ii) that it survives the first hour but decays in the second hour.

1.2 *Expected lifetime.* Let $L(a)$ denote the expected remaining lifetime at age a $(0 \leq a < \omega)$; in other words, $L(a)$ is the average time left until death for individuals who are alive and of age a. Show that $L(a) = \frac{1}{l(a)} \int_a^\omega l(t)dt$, where $l(a)$ is the fraction of individuals alive at age a and ω is the maximum possible age ($l(a) = 0$ for $a \geq \omega$).

1.3 *Expected lifetime with exponential decay.* Assume that the mortality rate $\mu > 0$ is constant.
(a) Show that the expected lifetime at birth is $L(0) = 1/\mu$.
IMPORTANT: this result will be used later in the course.
(b) Show that the expected remaining lifetime is independent of age, i.e., $L(a)$=L(0) for all $a \geq 0$. Explain verbally to a non-mathematician why this is so.

1.4. *Half-life.* For exponential decay ($\mu > 0$ constant), the half-life $t_{1/2}$ is defined such that after $t_{1/2}$ time, half of the original individuals (atoms, etc.) is still intact, i.e., $t_{1/2}$ is the solution of $l(t_{1/2})$=1/2. Show that the half-life is less than the expected lifetime (cf. exercise 1.3).

1.5 *Multiple modes of decay.* Suppose that an infected individual either dies (at a constant rate $\mu > 0$) or recovers from the disease (at a constant rate $\nu > 0$). What is the probability that he eventually recovers rather than dies?

To model the process of multiple exponential decay, denote the number of infected with $N(t)$ (where t is time since infection and $N(0) = N_0$ is given), the number of recovered with $R(t)$ and the number of dead with $D(t)$. Obviously, $R(0) = D(0) = 0$ and $N(t) + R(t) + D(t) = N_0$. Verify that the ODEs governing the dynamics of $N(t)$, $R(t)$ and $D(t)$

are

$$\frac{dN}{dt} = -(\mu + \nu) N(t) \tag{1}$$

$$\frac{dR}{dt} = \nu N(t) \tag{2}$$

$$\frac{dD}{dt} = \mu N(t) \tag{3}$$

Give the explicit solution for $R(t)$. Explain that he probability that an infected person eventually recovers is $lim_{t \to \infty} R(t)/N_0$, and show that this equals $\frac{\nu}{\mu+\nu}$.
IMPORTANT: this result will be used later in the course.