Normal forms

Introduction and notation

The text which follows aims at adding more details to the very concise discussion which can be found

- Chapter 2 of [1]

Self-consisten rigorous discussions of normal forms can be found in

- Chapter 3 of [3]
- Chapter 19 of [4]

Both presentations rely on results derived in the research paper [2], which is also a recommended reading.

As usual we suppose that

$$\dot{\boldsymbol{\phi}}_t = \boldsymbol{f} \circ \boldsymbol{\phi}_t \tag{0.1}$$

is driven by a vector field **sufficiently smooth** to guarantee the existence of a flow $\boldsymbol{\Phi} \colon \mathbb{R} \times \mathcal{D} \mapsto \mathcal{D}$ ($\mathcal{D}$ stand here as a generic symbol for the state space e.g. $\mathcal{D} = \mathbb{R}^n$) in terms of which we express the solution of (0.1) starting from $\boldsymbol{x}$ at time $t = 0$:

$$\boldsymbol{\phi}_t = \boldsymbol{\Phi}_t \circ \boldsymbol{x} \tag{0.2}$$

1 Normal form theory

The idea behind normal form theory is to construct a change of variables mapping a given ordinary differential equation driven by a non-linear vector field into one driven by a vector field as "close as possible" to linear. We won't use here the hyperbolicity assumption for the matrix A specifying the linear flow.

Concretely, let us suppose that we want to analyze the d-dimensional ordinary differential equation

$$\dot{\boldsymbol{\phi}}_{\boldsymbol{t}} = \boldsymbol{f}(\boldsymbol{\phi}_t) \equiv \mathsf{A} \cdot \boldsymbol{\phi}_t + \boldsymbol{g}(\boldsymbol{\phi}_t) \tag{1.1}$$

with

$$\boldsymbol{g}(\boldsymbol{0}) = 0 \qquad \& \qquad (\partial_{\boldsymbol{x}} \otimes \boldsymbol{g})(\boldsymbol{0}) = 0 \tag{1.2}$$

Were we able to find a change of variables mapping (1.1) into

$$\dot{\boldsymbol{\psi}}_{\boldsymbol{t}} = \mathsf{A} \cdot \boldsymbol{\psi}_t \tag{1.3}$$

at least locally in some neighborhood of the origin, we could gain quantitative insight into the flow solution of (1.1) by solving the linear problem (1.3).

As it will become clear in the following, in general it is not possible to find such a change of variables. We should instead more conservatively look for a change of variable mapping (1.1) into

$$\dot{\boldsymbol{\psi}}_{\boldsymbol{t}} = \mathsf{A} \cdot \boldsymbol{\psi}_t + \boldsymbol{n}(\boldsymbol{\psi}_t) \tag{1.4}$$

Here, the vector field $\boldsymbol{n}$ also enjoys the properties

$$\boldsymbol{n}(\mathbf{0}) = 0 \qquad \& \qquad (\partial_{\boldsymbol{x}} \otimes \boldsymbol{n})(\mathbf{0}) = 0 \tag{1.5}$$

Furthermore, we will define it "constructively" by reabsorbing into $\boldsymbol{n}$ all non-linear terms that we cannot remove performing a change of variables in (1.1).

Specifically, let $\mathcal{U}$ and $\mathcal{W}$ be neighborhoods of the origin. We look for a map

$$\boldsymbol{H} : \mathcal{U} \mapsto \mathcal{W} \tag{1.6}$$

such that for every t for which solutions of (1.1) exist we can write

$$\boldsymbol{\phi}_t = \boldsymbol{H}(\boldsymbol{\psi}_t) \equiv \boldsymbol{\psi}_t + \boldsymbol{h}(\boldsymbol{\psi}_t) \tag{1.7}$$

where $\boldsymbol{\psi}_t$ is solution of (1.4). We have therefore

$$\begin{aligned} &\dot{\boldsymbol{\psi}}_t \cdot \partial_{\boldsymbol{\psi}_t} \boldsymbol{H}(\boldsymbol{\psi}_t) = \\ &\qquad (\mathsf{A} \cdot \boldsymbol{\psi}_t) \cdot \partial_{\boldsymbol{\psi}_t} \boldsymbol{H}(\boldsymbol{\psi}_t) + \boldsymbol{n}(\boldsymbol{\psi}_t) \cdot \partial_{\boldsymbol{\psi}_t} \boldsymbol{H}(\boldsymbol{\psi}_t) = \mathsf{A} \cdot \boldsymbol{H}(\boldsymbol{\psi}_t) + \boldsymbol{g} \circ \boldsymbol{H}(\boldsymbol{\psi}_t) \end{aligned} \tag{1.8}$$

Since the equality must hold independently of t

$$\Big(\mathsf{A} \cdot \boldsymbol{x} + \boldsymbol{n}(\boldsymbol{x})\Big) \cdot \Big(1_d + \partial_{\boldsymbol{x}} \otimes \boldsymbol{h}(\boldsymbol{x})\Big) = \mathsf{A} \cdot \Big(\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})\Big) + \boldsymbol{g}(\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})) \tag{1.9}$$

We finally arrive to

$$\Big(1_d\, \boldsymbol{x} \cdot \mathsf{A}^{\mathsf{T}} \cdot \partial_{\boldsymbol{x}} - \mathsf{A}\Big) \cdot \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})) - \boldsymbol{n}(\boldsymbol{x}) - \boldsymbol{n}(\boldsymbol{x}) \cdot (\partial_{\boldsymbol{x}} \otimes \boldsymbol{h})(\boldsymbol{x}) \tag{1.10}$$

Note that in (1.10) there are two unknown vector fields: $\boldsymbol{h}$ and $\boldsymbol{n}$. In order to make sense of (1.10) we need to recall

i Setting $\boldsymbol{n} = 0$ yields the equation

$$\Big(1_d\, \boldsymbol{x} \cdot \mathsf{A}^{\mathsf{T}} \cdot \partial_{\boldsymbol{x}} - \mathsf{A}\Big) \cdot \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})) \tag{1.11}$$

specifying the vector field mapping (1.1) into a linear system.

ii Under our working hypotheses $\boldsymbol{g}$, $\boldsymbol{h}$, $\boldsymbol{n}$ are vector field vanishing *at least quadratically* at the origin.

iii We are interested in solving locally (1.10) locally around the origin. This means expanding all fields in Taylor series.

Based on the foregoing considerations, our solution scheme for (1.10) is to solve it by matching *order by order* the Taylor expansions of the left with the right hand side. In doing so, we set the coefficients of the expansion of $\boldsymbol{n}$ to zero whenever a solution can be found only in terms of the coefficients of $\boldsymbol{g}$ and $\boldsymbol{h}$. The outlined solution scheme specifies a well defined algorithm because we can show that every fixed order of the Taylor expansion can be treated independently. To prove this claim we need to gain some insight into the left and right hand sides of (1.10) independently.

1.1 The homological operator

We refer to the differential operator

$$\mathfrak{H}_{\boldsymbol{x}} = 1_d\,\boldsymbol{x}\cdot\mathsf{A}^{\mathsf{T}}\cdot\partial_{\boldsymbol{x}} - \mathsf{A} \tag{1.12}$$

as the "homological" operator. The name means "identical" (from the Greek $\grave{o}\mu\acute{o}\varsigma$, homos) and "rationale" ($\lambda\acute{o}\gamma o\varsigma$, logos) and reflects the following property

Proposition 1.1. *The homological operator (1.12) maps the space of differentiable homogeneous functions into itself*

Proof. Let $\lambda \in \mathbb{R}_+$ and f a differentiable function homogeneous of degree m. This means that

$$f(\lambda\,\boldsymbol{x}) = \lambda^m\,f(\boldsymbol{x}) \tag{1.13}$$

The dilation operator

$$\mathfrak{D}_{\boldsymbol{x}} = \boldsymbol{x}\cdot\partial_{\boldsymbol{x}} \tag{1.14}$$

acts on f as

$$\mathfrak{D}_{\boldsymbol{x}} f(\boldsymbol{x}) = m\,f(\boldsymbol{x}) \tag{1.15}$$

In words, $\mathfrak{D}_{\boldsymbol{x}}$ maps any differentiable homogeneous function into itself. A direct calculation shows that

$$[\mathfrak{D}_{\boldsymbol{x}}\,,\mathfrak{H}_{\boldsymbol{x}}] := \mathfrak{D}_{\boldsymbol{x}}\mathfrak{H}_{\boldsymbol{x}} - \mathfrak{H}_{\boldsymbol{x}}\mathfrak{D}_{\boldsymbol{x}} = 0 \tag{1.16}$$

This means that for any smooth homogeneous f of degree m

$$\mathfrak{D}_{\boldsymbol{x}}\mathfrak{H}_{\boldsymbol{x}} f(\boldsymbol{x}) = m\mathfrak{H}_{\boldsymbol{x}} f(\boldsymbol{x}) \tag{1.17}$$

which is equivalent to say that $\mathfrak{H}_{\boldsymbol{x}} f$ is homogeneous of degree m and yields therefore the proof of the proposition. □

This property of the homological operator is the cornerstone of the normal form method. Namely any individual order of the Taylor expansion of a vector field is specified by an homogeneous polynomial. In particular we can write the vector field $\boldsymbol{h}$ in (1.10) as

$$\boldsymbol{h}(\boldsymbol{x}) = \sum_{m=2}^{\infty}\frac{1}{m!}\sum_{i=1}^{d}\boldsymbol{e}_i\mathsf{H}^{(i)} : \overbrace{\boldsymbol{x}\otimes\cdots\otimes\boldsymbol{x}}^{m\,\text{times}} \tag{1.18}$$

where

- $\{\boldsymbol{e}_i\}_{i=1}^{d}$ is the canonical basis of $\mathbb{R}^d$

$$\boldsymbol{e}_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\qquad \boldsymbol{e}_2 = \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}\qquad \boldsymbol{e}_d = \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} \tag{1.19}$$

- $\left\{\mathsf{H}^{(i)}\right\}_{i=1}^{d}$ is a collection of d coefficients with m indices such that

$$(\mathsf{H}^{(i)} : \overbrace{\boldsymbol{x}\otimes\cdots\otimes\boldsymbol{x}}^{m\,\text{times}}) \equiv \left(\sum_{j_1,\dots,j_m=1}^{d}\mathsf{H}^{(i)}_{j_1,\dots,j_m}x^{j_1}x^{j_2}\dots x^{j_m}\right) : \mathbb{R}^d \mapsto \mathbb{R} \qquad \forall\, i = 1,\dots,d \tag{1.20}$$

Thus the m-th order of the Taylor expansion of $\boldsymbol{h}$ can be written as

$$[\boldsymbol{h}(\boldsymbol{x})]_m = \frac{1}{m!}\sum_{i=1}^{d} \boldsymbol{e}_i \mathsf{H}^{(i)} : \overbrace{\boldsymbol{x}\otimes\cdots\otimes\boldsymbol{x}}^{m\,\text{times}} \tag{1.21}$$

where the right hand side is a linear combinations of vector polynomials of $\boldsymbol{x}\in\mathbb{R}^d$ homogeneous of degree m:

$$[\boldsymbol{h}(\lambda\,\boldsymbol{x})]_m = \lambda^m\,[\boldsymbol{h}(\boldsymbol{x})]_m \tag{1.22}$$

Example 1.1. Let

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1.23}$$

and

$$\boldsymbol{P}(\boldsymbol{x}) = \begin{bmatrix} p_{11}^{(1)} x_1^2 + 2\,p_{12}^{(1)} x_1 x_2 + p_{22}^{(1)} x_2^2 \\ p_{11}^{(2)} x_1^2 + 2\,p_{12}^{(2)} x_1 x_2 + p_{22}^{(2)} x_2^2 \end{bmatrix} = \sum_{i=1}^{2} \boldsymbol{e}_i \left(p_{11}^{(i)} x_1^2 + p_{12}^{(i)} x_1 x_2 + p_{22}^{(i)} x_2^2 \right)$$

or equivalently using the notation adopted in (1.18)

$$\boldsymbol{P}(\boldsymbol{x}) = \sum_{i=1}^{2} \boldsymbol{e}_i \mathsf{P}^{(i)} : \boldsymbol{x}\otimes\boldsymbol{x} \tag{1.24}$$

where for $i = 1, 2$

$$\mathsf{P}^{(i)} = \begin{bmatrix} p_{11}^{(i)} & p_{12}^{(i)} \\ & \\ p_{12}^{(i)} & p_{22}^{(i)} \end{bmatrix} \qquad \Rightarrow \qquad \mathsf{P}_{11}^{(i)} = p_{11}^{(i)} \quad \& \quad \mathsf{P}_{12}^{(i)} = \mathsf{P}_{21}^{(i)} = p_{12}^{(i)} \quad \& \quad \mathsf{P}_{22}^{(i)} = p_{22}^{(i)} \tag{1.25}$$

Let $\mathcal{H}_m(\mathbb{R}^d;\mathbb{R}^d)$ the space of d-dimensional vector homogeneous polynomials of degree m of a d-dimensional variable. Then we have

$$[\boldsymbol{h}(\boldsymbol{x})]_m \in \mathcal{H}_m(\mathbb{R}^d;\mathbb{R}^d) \tag{1.26}$$

and, since the homological operator maps homogeneous polynomials of degree m into homogeneous polynomials of degree m, also

$$\mathfrak{H}_{\boldsymbol{x}}[\boldsymbol{h}(\boldsymbol{x})]_m \in \mathcal{H}_m(\mathbb{R}^d;\mathbb{R}^d) \tag{1.27}$$

This observation implies that we can write the left hand side of (1.10) as

$$\mathfrak{H}_{\boldsymbol{x}}\boldsymbol{h}(\boldsymbol{x}) = \sum_{m\geq 2} \mathfrak{H}_{\boldsymbol{x}}[\boldsymbol{h}(\boldsymbol{x})]_m \tag{1.28}$$

where each addend in the series lives in in a different $\mathcal{H}_m(\mathbb{R}^d;\mathbb{R}^d)$.

1.1.1 Spectral properties of the homological operator

Let us focus the attention on the homological operator $\mathfrak{H}_{\boldsymbol{x}}$ when acting on $\mathcal{H}_m(\mathbb{R}^d;\mathbb{R}^d)$ for any fixed m. Then $\mathfrak{H}_{\boldsymbol{x}}$ is a linear operator. Furthermore let us suppose that the matrix A in (1.10) is *diagonal*:

$$\mathsf{A} = \sum_{i=1}^{d} a_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_d \end{bmatrix} \tag{1.29}$$

Then we can couch the homological operator into the simpler form

$$\mathfrak{H}_{\boldsymbol{x}} = 1_d \sum_{j=1}^{d} a_j x_j \partial_{x_j} - \sum_{j=1}^{d} a_j \, \wp_j \tag{1.30}$$

where $\wp_j$ is the projector

$$\wp_j \boldsymbol{e}_i = \delta_{ij} \boldsymbol{e}_i \tag{1.31}$$

We can then prove that

Proposition 1.2. *The homological operator is diagonal on homogeneous vectors of the form*

$$\boldsymbol{v}(\boldsymbol{m}, i) = \boldsymbol{e}_i \prod_{j=1}^{d} x_j^{m_j} \tag{1.32}$$

where $\boldsymbol{m} = [m_1, \dots, m_d]$ are integer numbers satisfying the constraint

$$m = \sum_{i=1}^{d} m_i \tag{1.33}$$

where m is the degree of homogeneity of the polynomials in $\mathcal{H}_m(\mathbb{R}^d;\mathbb{R}^d)$. Furthermore,

$$\mathfrak{H}_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{m}, i) = \Lambda(\boldsymbol{m}, i) \, \boldsymbol{v}(\boldsymbol{m}, i) \tag{1.34}$$

with

$$\Lambda(\boldsymbol{m}, i) = \sum_{j=1}^{d} m_j \, a_j - a_i \tag{1.35}$$

Proof. By direct calculation we see that

$$1_d \sum_{j=1}^{d} a_j x_j \partial_{x_j} \boldsymbol{v}(\boldsymbol{m}, i) = \sum_{j=1}^{d} a_j \, m_j \boldsymbol{v}(\boldsymbol{m}, i) \tag{1.36}$$

and

$$\sum_{j=1}^{d} a_j \wp_j \boldsymbol{v}(\boldsymbol{m}, i) = a_i \, \boldsymbol{v}(\boldsymbol{m}, i) \tag{1.37}$$

□

It turns out that it is possible to introduce in $\mathcal{H}_m(\mathbb{R}^d;\mathbb{R}^d)$ a suitable scalar product and to show that the homological operator has a non-empty null-space or kernel if and only if the resonance condition

$$\Lambda(\boldsymbol{m}, i) = 0 \qquad \& \qquad \sum_j m_j = m \qquad m_j \in \mathbb{N} \quad \forall j \tag{1.38}$$

is verified. We refer to the span (i.e. the set of all possible linear combinations) of the eigenvectors of $\mathfrak{H}_{\boldsymbol{x}}$ with zero eigenvalue as the *kernel of* $\mathfrak{H}_{\boldsymbol{x}}$ in $\mathcal{H}_m(\mathbb{R}^d;\mathbb{R}^d)$. We denote this set as $\mathrm{Ker}\mathfrak{H}\mathcal{H}_m$ The mentioned scalar product was introduced in [2].

1.2 Analysis of the right hand side of the homological equation (1.10)

Let us first set $\boldsymbol{n} = 0$ into (1.10) or equivalently consider (1.11). Adopting for $\boldsymbol{g}$ the same notation as in (1.18):

$$\boldsymbol{g}(\boldsymbol{x}) = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{i=1}^{d} \boldsymbol{e}_i \mathsf{G}^{(i)} : \overbrace{\boldsymbol{x} \otimes \cdots \otimes \boldsymbol{x}}^{m\,\text{times}} = \sum_{m=2}^{\infty} [\boldsymbol{g}(\boldsymbol{x})]_m \tag{1.39}$$

it follows immediately that

$$[\boldsymbol{g}(\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x}))]_m = \sum_{i=1}^{d} \frac{\boldsymbol{e}_i}{m!} \mathsf{G}^{(i)} : \overbrace{\Big(\boldsymbol{x} + \boldsymbol{h}_m(\boldsymbol{x}) \otimes \cdots \otimes \boldsymbol{x} + \boldsymbol{h}_m(\boldsymbol{x}) \Big)}^{m\,\text{times}} = [\boldsymbol{g}(\boldsymbol{x})]_m +$$

$$\sum_{i=1}^{n} \frac{\boldsymbol{e}_i}{(m-1)!} \mathsf{G}^{(i)} : \boldsymbol{h}(\boldsymbol{x}) \otimes \overbrace{\boldsymbol{x} \otimes \cdots \otimes \boldsymbol{x}}^{m-1\,\text{times}} + \cdots + \sum_{i=1}^{n} \frac{\boldsymbol{e}_i}{m!} \mathsf{G}^{(i)} : \overbrace{\boldsymbol{h}(\boldsymbol{x}) \otimes \boldsymbol{h}(\boldsymbol{x}) \otimes \cdots \otimes \boldsymbol{h}(\boldsymbol{x})}^{m\,\text{times}} \tag{1.40}$$

We now recall that

$$\boldsymbol{h}(\boldsymbol{x}) = O(\|\boldsymbol{x}\|^r) \qquad r \geq 2 \tag{1.41}$$

which means that

$$\boldsymbol{g}(\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x}))]_m = [\boldsymbol{g}(\boldsymbol{x})]_m + o(\|\boldsymbol{x}\|^n) \tag{1.42}$$

In words, all the terms depending upon at least one insertion of $\boldsymbol{h}$ are always higher order in $\boldsymbol{x}$ than $[\boldsymbol{g}(\boldsymbol{x})]_m$.

1.3 Solution of the homological equation

We are in the position to prove

Proposition 1.3. *For any $r \in \mathbb{N}$ there exists, in a neighborhood of the origin, a polynomial change of variables induced by the formal diffeomorphism*

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(y) = \boldsymbol{y} + \sum_{k=2}^{r} \boldsymbol{h}(y) + O(\| \boldsymbol{y} \|^{r+1}) \tag{1.43}$$

with $\boldsymbol{h}_k \in \mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)$ mapping (1.1) into

$$\dot{\boldsymbol{\psi}}_t = \mathsf{A} \cdot \boldsymbol{\psi}_t + \sum_{k=2}^{r} \boldsymbol{n}_k(\boldsymbol{\psi}_t) + O(\| \boldsymbol{\psi}_t \|^{r+1}) \tag{1.44}$$

where

$$\boldsymbol{n}_k(\boldsymbol{x}) \in \mathrm{Ker}\mathfrak{H}\mathcal{H}_m \qquad \forall\, k \;|\; 2 \leq k \leq r \tag{1.45}$$

Proof. The proof is constructive and proceeds by induction. Here we present only the main idea. Let us suppose that

$$\boldsymbol{g}(\boldsymbol{x}) = O(\|\boldsymbol{x}\|^p) \qquad p \geq 2 \tag{1.46}$$

We know that a resonance occur whenever

$$\Lambda(\boldsymbol{m}, i) = 0\,, \qquad m = \sum_{j=1}^{d} m_j \geq p \tag{1.47}$$

The knowledge of $\boldsymbol{m}$ and i specify modulo a proportionality which must be determined by direct calculation the non-vanishing contributions to $\boldsymbol{n}$.

- if we retain only leading order terms, (1.10) becomes

  $$\mathfrak{H}_{\boldsymbol{x}}\,[\boldsymbol{h}(\boldsymbol{x})]_p = [\boldsymbol{g}(\boldsymbol{x})]_p - [\boldsymbol{n}(\boldsymbol{x})]_p + o(\|\boldsymbol{x}\|^p) \tag{1.48}$$

  We distinguish two sub-cases.

  1. if (1.47) does not admit solution for $m = p$ we set $\boldsymbol{n}$ to zero and we look for the unique solution of

     $$\mathfrak{H}_{\boldsymbol{x}}\,[\boldsymbol{h}(\boldsymbol{x})]_p = [\boldsymbol{g}(\boldsymbol{x})]_p \tag{1.49}$$

  2. if (1.47) admits a solution for some $(\boldsymbol{m}, i)$ we choose $[\boldsymbol{n}(\boldsymbol{x})]_p$ such to subtract the corresponding terms from $\boldsymbol{g}$ and then we look for the unique solution of

     $$\mathfrak{H}_{\boldsymbol{x}}\,[\boldsymbol{h}(\boldsymbol{x})]_p = [\boldsymbol{g}(\boldsymbol{x})]_p - [\boldsymbol{n}(\boldsymbol{x})]_p \tag{1.50}$$

- Suppose first that we have found a change of variables mapping (1.1) into

  $$\dot{\boldsymbol{\psi}}_t = \mathsf{A} \cdot \boldsymbol{\psi}_t + \sum_{k\geq 2}^{r-1} [\boldsymbol{n}(\boldsymbol{\psi}_t)]_k + [\boldsymbol{g}(\boldsymbol{\psi}_t)]_r + O(\|\ \boldsymbol{\psi}_t\ \|^{r+1}) \tag{1.51}$$

  where, as usual $[\boldsymbol{n}]_k$ is an homogeneous polynomial of degree k and $[\boldsymbol{g}]_r$ is the homogeneous polynomial of degree r specifying the r-th order of the Taylor expansion of $\boldsymbol{g}$.

- Observe that a change of variables of the form

  $$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}_r(\boldsymbol{y}) \tag{1.52}$$

  with $\boldsymbol{h}_r = O(\|\boldsymbol{x}\|^r)$ yields

  $$\begin{aligned} &\mathsf{A} \cdot [\boldsymbol{y} + \boldsymbol{h}_r(\boldsymbol{y})] + \sum_{k=2}^{r-1} \boldsymbol{n}_k(\boldsymbol{y} + \boldsymbol{h}_r(\boldsymbol{y})) \\ &\quad = \mathsf{A} \cdot \boldsymbol{y} + \mathsf{A} \cdot \boldsymbol{h}_r(\boldsymbol{y}) + \sum_{k=2}^{r-1} \boldsymbol{n}_k(\boldsymbol{y}) + \sum_{k=2}^{r-1} O(\|\ \boldsymbol{x}\ \|^{m+k-1}) \end{aligned} \tag{1.53}$$

  In words: resonant terms identified by solving the homological equation up to order $r-1$ in Taylor expansion are not affected by a change of variables of the form (1.52). This means that we can again restrict the analysis to the solution of

  $$\mathfrak{H}_{\boldsymbol{x}}\,[\boldsymbol{h}(\boldsymbol{x})]_r = [\boldsymbol{g}(\boldsymbol{x})]_r - [\boldsymbol{n}(\boldsymbol{x})]_r \tag{1.54}$$

  with $-[\boldsymbol{n}(\boldsymbol{x})]_r$ non vanishing only if (1.47) admits solutions at the order r

□

2 Normal forms of the anharmonic oscillator

The equations for the anharmonic oscillator in non-dimensional coordinates are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \varepsilon\, x_1^3 \end{aligned} \tag{2.1}$$

If we adopt the vector notation (1.1) we have

$$\mathsf{A}\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \& \qquad \boldsymbol{g}(\boldsymbol{x}) = -\varepsilon \begin{bmatrix} 0 \\ x_1^3 \end{bmatrix} \tag{2.2}$$

The eigenvalues of A are $a = \pm\imath$.

It is convenient to pass preliminarily to a system of coordinates where the linear part is diagonal. To this end we set

$$z = x_1 + \imath\, x_2 \qquad \& \qquad \bar{z} = x_1 - \imath\, x_2 \tag{2.3}$$

The new equations of motion are

$$\begin{aligned} \dot{z} &= -\imath\, z - \varepsilon\, \imath \left(\frac{z+\bar{z}}{2}\right)^3 \\ \dot{\bar{z}} &= \imath\, \bar{z} + \varepsilon\, \imath \left(\frac{z+\bar{z}}{2}\right)^3 \end{aligned} \tag{2.4}$$

In the basis where A is diagnal we can easily identify the terms associated to resonances. Namely we see that the first resonances are encountered for

$$\Lambda((2,1),1) = 0 \qquad \Rightarrow \qquad \boldsymbol{v}((2,1),1) = \boldsymbol{e}_1 z^2\, \bar{z} \tag{2.5}$$

and

$$\Lambda((1,2),2) = 0 \qquad \Rightarrow \qquad \boldsymbol{v}((2,1),1) = \boldsymbol{e}_2 z\, \bar{z} \tag{2.6}$$

Hence we expect that a change of variables

$$\begin{bmatrix} z \\ \bar{z} \end{bmatrix} = \begin{bmatrix} y \\ \bar{y} \end{bmatrix} + \begin{bmatrix} h_{30}^{(1)} y^3 + h_{21}^{(1)} y^2 \bar{y} + h_{12}^{(1)} y \bar{y}^2 + h_{03}^{(1)} \bar{y}^3 \\ h_{30}^{(2)} y^3 + h_{21}^{(2)} y^2 \bar{y} + h_{12}^{(2)} y \bar{y}^2 + h_{03}^{(1)} \bar{y}^3 \end{bmatrix} + \text{h.o.t.} \tag{2.7}$$

yields

$$\begin{aligned} \dot{y} &= -\imath\, y - \frac{3\,\varepsilon\,\imath}{8}\, y^2\, \bar{y} + \text{h.o.t.} \\ \dot{\bar{y}} &= \imath\, \bar{y} + \frac{3\,\varepsilon\,\imath}{8}\, y\, \bar{y}^2 + \text{h.o.t.} \end{aligned} \tag{2.8}$$

The meaning of these equation is best understood adopting the polar representation of complex coordinates

$$z = |z|\, e^{\imath\,\theta} \tag{2.9}$$

yields away from the origin

$$\dot{|z|} = 0 \tag{2.10}$$

and

$$\dot{\theta} = -\left(1 + \frac{3\,\varepsilon}{8}|y|^2\right) \tag{2.11}$$

THe relation uncovers the dependence of the period on the amplitude for non-linear oscillations. In particular

$$|\theta| = 2\,\pi \tag{2.12}$$

implies

$$T = \frac{2\,\pi}{1 + \frac{3\,\varepsilon}{8}|y|^2} = 2\pi\left(1 - \frac{3\,\varepsilon}{8}|y|^2\right) + O(\varepsilon^2) \tag{2.13}$$

where $|y|$ is the amplitude of the oscillation.

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