INTRODUCTION TO DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 8

P. MURATORE-GINANNESCHI, K. SCHWIEGER

1. Poincaré Lindstedt method

In this exercise we consider the Duffing equation $\ddot{x} + x + \varepsilon x^3 = 0$ for small values of ε and fixed initial conditions $x(0) = a$ and $\dot{x}(0) = 0$. (You may be have seen this system before.)

(1) Show that the solutions of of the equation are periodic.

In this exercise we are going to find an approximate formula for the solution. Here, since we know that the solution is periodic, we will see that we can use regular perturbation theory. For this purpose let ω be the (unknown) frequency, which also depends on ε.

(1) Show that in the rescaled time $\tau := \omega t$ the equation reads $\omega^2 x'' + x + \varepsilon x^3 = 0$.
(2) Suppose that x and τ admit an expansion in ε in *regular* perturbation theory, i. e.,

$$x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \dots, \qquad \omega(\varepsilon) = 1 + \varepsilon\omega_1 + \dots.$$

Find differential equations for the coefficients up to the first non-trivial order.
(3) Solve the eqation for x_0 using the initial conditions.
(4) Solve the equation for x_1 by assuming that the secular terms vanish. What is ω_1?

2. Prototypes

For the following prototype examples, sketch the phase portrait for various (crucial) values of μ

(1) $\dot{x} = \mu x - x^2$, $\dot{y} = -y$,
(2) $\dot{x} = \mu x - x^3$, $\dot{y} = -y$,
(3) $\dot{x} = \mu x + x^3$, $\dot{y} = -y$.

3. A Bifurcation

Find and classify all bifurcations for the system $\dot{x} = y - ax$, $\dot{y} = -by + x/(1 + x)$.

4. Zero Eigenvalue Bifurcation

Consider a bifurcation of a two-dimensional system where eigenvalues turn from non-zero to at least one zero eigenvalue. Show that at the bifurcation point the nullclines intersect tangentially.

5. Normal Form

Consider (again) the Duffing equation $\ddot{x} + x + \mu x^3 = 0$ for some $\mu \in \mathbb{R}$. Determine the normal form of order 3 around the origin.

Date: 6th November 2015.