INTRODUCTION TO DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 5

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1. Higher-Order Fixed Points

Consider the system

$$\dot{x} = xy - x^2 y + y^3, \qquad \dot{y} = y^2 + x^3 - xy^2$$

Show that the origin is the only fixed point. Determine its stability and sketch the phase portrait.
Hint: Polar coordinates may simplify the equations a bit.

2. A Class of Reversible Systems

Consider the equation

$$\ddot{x} + f(\dot{x}) + g(x) = 0.$$

with smooth functions f and g. Show that the equation has the time-reversal symmetry $t \mapsto -t$ if f if the function f is even. Conclude that fixed points cannot be stable nodes or spirals.

3. Conservative Systems

For the following systems, find a conserved quantity. Find the fixed points and sketch the phase portrait.

(1) $\ddot{x} = x^3 - x$, (2) $\ddot{x} = x - x^2$, (3) $\ddot{x} = a - e^x$ for $a \in \mathbb{R}$.

4. Non-Isolated Fixed Points

Consider the system $\dot{x} = xy$, $\dot{y} = -x^2$.

(1) Show that $E(x, y) := x^2 + y^2$ is a conserved quantity.
(2) Show that although E has a local miminum at the origin, but it is not a center fixed point. Sketch the phase portrait.

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5. The Mantra Ray and Symmetry

Consider the system

$$\dot{x} = y - y^3, \qquad \dot{y} = -x - y^2.$$

You may want to use the computer first to get an idea about the vector field. In this exercise we will actually prove facts that you may immediately see or verify from the plot.

(1) Compute the nullclines.
(2) Determine the sign of $\dot{x}$ and $\dot{y}$ in different regions of the plane.
(3) What happens under the transformation $y \mapsto -y$ and what does it mean to the vector field?
(4) Compute the fixed points, determine their type. For the sattle points determine the eigendirections of the flow.
(5) Prove that the stable manifold (leading towards the fixed point) of $(-1, 1)$ intersects the x-axis with a negative y-coordinate. Use the symmetry to deduce that existence fo a heteroclinit trajectory between $(-1, 1)$ and $(-1, -1)$.
(6) Show that there is another heteroclinic trajectory connecting $(-1, 1)$ and $(-1, -1)$.