INTRODUCTION TO DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 3

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1. Ising Model

A simple, prominent model for a magnet is the so called *Ising model*. It is especially famous for its phase transition. In this model a magnet consists of a large number $N \gg 1$ of so called spins. Each spin can point only up or down. We write $S_i = \pm 1$ depending on the direction of the spin. The individual spins are arranged in a regular lattice. At this point we do not specify the precise full dynamics of the model but rather roughly describe its main three effects.

(A) The first effect is due to the interactions among the spins. Every spin interacts (pairwise) with its neighboring spins in such a way that it is energetically favourable for the spins to be aligned, that is, both pointing up or both pointing down. The strength of this effect is determined by a constant $J > 0$, the *coupling strength*.

(B) The second effect is due to a magnetic field in which the magnet is located. Depending on the direction of the field each spin energetically favours either the up or the down state. The strength of this effect is determined by the *field strength* $h > 0$.

(C) The third effect is due to thermal fluctuations. At nonzero temperature a spin may randomly flip its orientation. The larger the temperature the more likely is such a flip or, equivalently, the more frequent such a flip occurs. The strength of this effect is is determined by the temperature $T \geq 0$.

The quantity that we are interested in is the total *magnetization*

$$m = \left| \frac{1}{N} \sum_{i=1}^{N} S_i \right|$$

Before we go to the dynamical system, let us get familiar with the physics a bit.

(1) To get familiar with the physics, can you guess stationary states if only one (or two) of the effects but not the others are active. For instance, what happens without a magnetic field at zero temperature.

In a reasonable approximation of the involved dynamics, the magnetization satisfies the equation

$$h = T \cdot \tanh^{-1}(m) - Jm$$

(1) Analyze the solutions m of the above equation graphically.

(2) Suppose there is no magenetic field, i. e., $h = 0$. Find the critical temperature T_c at which a bifurcation happens. What does it mean for the magnet?

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2. Singular Limits

Consider the equation $\varepsilon\ddot{x} + \dot{x} + x = 0$ with the initial condition $x(0) = 1$ and $\dot{x}(0) = 0$.

(1) Solve the system analytically. (The system is linear.)
(2) Suppose $0 < \varepsilon \ll 1$. Show that there are two widely separated time scales in the problem. Estimate them in term of ε.
(3) Look at the graph of the solution for small ε. Do you see the time scales?
(4) What do you think about replacing the above eqution by $\dot{x} + x = 0$?

3. Periodic Solutions, Part 1

Show that a differential equation $\dot{x} = f(x)$ on the line cannot have periodic solutions except for constant solutions.

4. Periodic Solutions, Part 2

Whereas there are no non-trivial dynamical systems on the line, there are such systems on the circle. For this exercise it is convenient to consider the circle as the quotient $\mathbb{S} = \mathbb{R}/(2\pi\mathbb{Z})$. That is, we parametrize the circle by an angle $\theta \in \mathbb{R}$ with the convention that θ, $\theta \pm 2\pi$, $\theta \pm 4\pi$, etc. specify the same point.

(1) Consider the equation $\dot{\theta} = \omega$ with a constant $\omega \in \mathbb{R}$. How do solutions look like and what is the role of ω?
(2) Consider the equation $\dot{\theta} = \sin\theta$. Find all fixed points and determine their stability.
(3) Consider the equation $\dot{\theta} = \omega - a\sin\theta$ with $\omega > 0$ and $a \geq 0$. Determine graphically the number of fixed points and their stability depending on the value of ω and a. Sketch a bifurcation diagram.