INTRODUCTION TO DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 2

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1. Potentials and Bifurcations

For each of the following examples, sketch the potentials for different values of r. Be sure to show all qualitatively different cases, including bifurcation values.

(1) $\dot{x} = r - x^2$, (2) $\dot{x} = rx - x^2$, (3) $\dot{x} = rx + x^3 - x^5$,

2. Type of a Bifurcation

Consider the following systems with a parameter r. For each system determine the crticial point and qualitatively sketch the different vector fields that occur as r is varied. Sketch the bifurcation diagram (fixed points versus r.

(1) $\dot{x} = 1 + rx + x^2$, (2) $\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$, (3) $\dot{x} = rx + x^2$,
(4) $\dot{x} = x - rx(1-x)$, (5) $\dot{x} = x(r - e^x)$, (6) $\dot{x} = x + \tanh(rx)$,
(7) $\dot{x} = rx - \frac{x}{1+x}$, (8) $\dot{x} = rx - \frac{x}{1+x^2}$, (9) $\dot{x} = rx + \frac{x^3}{1+x^2}$.

3. A More Interesting Example

Consider the system $\dot{x} = rx - \sin x$.

(1) Sketch the bifurcation diagram without classifying the bifurcations.
(2) Classify all bifurcations that occur for $r > 0$.
(3) For $0 < r \ll 1$, find an approximate formula for values of r at which bifurcations occur.
(4) Describe the stability of the fixpoints that occur for $r < 0$.

4. Logistic Equation Revised, Fishery

During the lecture you have seen the logistic equation $\dot{N} = rN(1 - N/K)$ with constants $r, K > 0$ as a simple model of population grows.

(1) Recall the relation of the constants r, K and the population groth model. What do r and K describe?
(2) If the population is harvested (fishery) with a constant rate, we may instead consider the equation $\dot{N} = rN(1 - N/K) - H$ with an additional constant $H > 0$. Show that by a suitable transformation of variables the system is equivalent to
$$\dot{x} = x(1 - x) - h. \qquad (1)$$
Determine and classify the bifurcations of equation (1).
(3) Why is this model not satisfying or what effect of the equation does not match reality?

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(4) A more refined model would be to consider the equation

$$\det N = rN\Big(1 - \frac{N}{K}\Big) - \frac{HN}{A+N} \qquad (2)$$

with constants $A, H > 0$. How does the "harvest term" $\frac{HN}{A+N}$ behave in dependence on N and why is this a better model?

(5) Show that by a suitable change of variables the system (2) can be reduced to

$$\dot{x} = x(1-x) - \frac{hx}{a+x}.$$

with $a, h > 0$. Find and classify the fixed points for each choice of a and h. What kind of stability do you find in the different regions?