INTRODUCTION TO
DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 1

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Before you start

There will be multiple group exercises in this week. So before you start: Find a group or a partner to work with.

1. Warm-Up

How are the equations $\dot{x} = f(x)$ and $\dot{x} = -f(x)$ related. More precisely, how does the phase diagram and how do the solutions relate to each other?

2. Differential Calculus on the Line

(1) Ask your partner to draw the graph of a function f. Provide him with the figure of the derivatives f' and f''.
(2) Conversely, provide your partner with the graph of a function f. Ask him to figure functions g with $g' = f$ and h with $h'' = f$.

3. Stability

Consider the following dynamical systems on the line:

(1) $\dot{x} = 4x^2 - 16$, (2) $\dot{x} = 1 - x^{14}$, (3) $\dot{x} = x - x^3$,
(4) $\dot{x} = e^{-x} \cdot \sin x$, (5) $\dot{x} = e^x - \cos x$.

Work first graphically and then analytically. Find the fixed points. Which of them are stable (unstable)? Sketch the solution for different initial values.

4. Working Backwards

(1) Ask your partner to draw a phase portrait. Provide a corresponding system equation $\dot{x} = f(x)$ that is consistent with the given phase portrait.
(2) For a given phase portrait, describe all functions f that lead to the given phase portrait. What kind of manifold is formed by this functions?
(3) Ask your partner to draw a solutions with different initial values. Provide a corresponding system equation $\dot{x} = f(x)$ that is consistent with the given solutions.
(4) How do you identify fixed points and their stability from the solutions? In particular, how many fixed points are there?
(5) Return the favour to your partner and provide him a few solutions for different initial values. But this time make sure there are as many types of stabilities as possible.

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5. The Logistic Equation

Consider the logicstic equation

$$\dot{N} = rN(1 - N/K)$$

for constants $r, K > 0$. Solve this equation analytically.
Hint: Look at $1/N$.

6. Finite Time

Consider a particle on the half-line $[0, \infty[$ moving according to the equation $\dot{x} = -x^c$ for some constant $c \in \mathbb{R}$.

(1) Determine all values of c for which the origin is a stable fixed point.
(2) Assume that c is chosen such that $x = 0$ is stable. How long it take for the particle to move from $x = 1$ to $x = 0$?

7. No Analytic Solution

Consider the equation $\dot{x} = x + e^{-x}$ with the initial condition $x(0) = 0$.

(1) Sketch the solution $x(t)$ for $t \geq 0$.
(2) Find bound for the value $x(1)$, that is, find $a, b \in \mathbb{R}$ with $a < x(1) < b$.