Introduction to Continuous Logic
Department of Mathematics and Statistics, University of Helsinki
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Exercise 4

1. Prove the quantifier case in the inductive proof of the Fundamental Theorem of Ultraproducts.

2. Dacunha-Castelle and Krivine introduced ultraproducts of Banach spaces in the 1970s. They started with a collection $(E_i)_{i\in I}$ of Banach spaces, then considered the bounded elements from their Cartesian product

$$\ell_\infty(I,E_i) = \{(x_i)_{i\in I} \in \prod_{i\in I} E_i : \sup_{i\in I} \|x_i\|_{E_i} < \infty\}.$$

This is a Banach space under componentwise addition and scalar multiplication. For D an ultrafilter on I, let

$$N_D = \{(x_i)_{i\in I} \in \ell_\infty(I,E_i) : \lim_{i,D} \|x_i\|_{E_i} = 0\}$$

which is a closed linear subspace of $\ell_\infty(I,E_i)$. The *Banach space ultraproduct* of the E_i with respect to D is defined as

$$(E_i)_D = \ell_\infty(I,E_i)/N_D.$$

Show that if M_i is the unit ball of E_i, then the ultraproduct $(\prod_{i\in I} M_i)_D$ defined on this course is the unit ball of the Banach space ultraproduct $(E_i)_D$ introduced by Dacunha-Castelle and Krivine.

3. Chapter 17 of the course material sketches how to axiomatize atomless L^p Banach lattices. Show that ℓ^p-spaces (atomic L^p-spaces) cannot be axiomatized with continuous logic. [Hint: being atomic cannot be axiomatized; use the compactness theorem.]

4. Show that compactness is not axiomatizable in continuous logic.

5. Proposition 7.3 of the material (Downward Löwenheim-Skolem Thm) shows how to find an elementary substructure of given size. Prove the Upward Löwenheim-Skolem Theorem: Let $\mathcal{M}$ be a non-compact L-structure and $\kappa > \text{density}(M)$. Then there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that $\text{density}(N) = \kappa$.