HARMONIC ANALYSIS AND SQUARE FUNCTIONS: EXERCISE SET 3

Return your written solutions preferably directly to Emil Vuorinen (office C435, emil.vuorinen@helsinki.fi). You can also return them during the lectures. **Deadline for the third set is Friday, December** 4**.** Ask for hints!

(1) Check that the martingales (defined in the p. 30 of the lecture notes) $\Delta_P f$, $P \in \mathcal{D}^{\mathrm{tr}}$, satisfy $\int_P \Delta_P f \, d\sigma = 0$.

(2) Let $(s_t)_{t>0}$ be an m-LP-family and V be the corresponding vertical square function. Let $Q \subset \mathbb{R}^n$ be a cube and σ be a finite Radon measure so that $\operatorname{spt} \sigma \subset Q$. Suppose that there is a Borel set $E \subset \mathbb{R}^n$ so that $\sigma(E) > 0$ and for every $x \in E$ there holds that

$$V_{\sigma,Q} 1_Q(x) < \infty \qquad \text{and} \qquad \sup_{r>0} \frac{\sigma(B(x,r))}{r^m} < \infty.$$

Show that there exists $G \subset E$ and $C < \infty$ so that $\sigma(G) > 0$ and

$$\|1_G V_{\sigma,Q} f\|_{L^2(\sigma)} \leq C \|f\|_{L^2(\sigma)}$$

for every $f \in L^2(\sigma)$.

(3) Let $(s_t)_{t>0}$ be an m-LP-family, $Q \subset \mathbb{R}^n$ be a cube and μ be a Radon measure in $\mathbb{R}^n$. Show that for every $f \in L^2(\mu)$ for which $\operatorname{spt} f \subset Q$ we have

$$\Big\| x \mapsto 1_Q(x) \Big(\int_{\ell(Q)}^{\infty} |\theta_t^\mu f(x)|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{L^2(\mu)} \leq C \frac{\mu(Q)}{\ell(Q)^m} \|f\|_{L^2(\mu)}.$$

(4) Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two dyadic systems in $\mathbb{R}^n$. Let $\gamma \in (0,1)$ and $r \geq 1$. Like in the lecture notes we say that $Q \in \mathcal{D}_1$ is (γ, r)-$\mathcal{D}_2$-good (or just $\mathcal{D}_2$-good), if $d(Q, \partial R) > \ell(Q)^\gamma \ell(R)^{1-\gamma}$ for every $R \in \mathcal{D}_2$ satisfying that $\ell(R) \geq 2^r \ell(Q)$. We denote these cubes here by $\mathcal{D}_{1,\text{good}}$.

Let μ be a radon measure in $\mathbb{R}^n$ and $M > 1$. We say that $a \in \mathrm{BMO}^2_M(\mu)$ if a is locally integrable and there is constant $C < \infty$ so that

$$\Big(\int_L |a - \langle a \rangle_L|^2 \, d\mu \Big)^{1/2} \leq C \mu(ML)^{1/2}$$

for every cube $L \subset \mathbb{R}^n$. Here

$$\langle a \rangle_L = \langle a \rangle_L^\mu = \frac{1}{\mu(L)} \int_L a \, d\mu.$$

The best constant C is denoted by $\|a\|_{\mathrm{BMO}^2_M(\mu)}$.

Recall also the definition of the standard martingales from the lecture notes

$$D_Q a = \sum_{Q' \in \mathrm{ch}(Q)} [\langle a \rangle_{Q'} - \langle a \rangle_Q] 1_{Q'}, \qquad Q \in \mathcal{D}_1.$$

We define the operator

$$\Pi_a f = \sum_{R \in \mathcal{D}_2} \sum_{\substack{Q \in \mathcal{D}_{1,\mathrm{good}} \\ Q \subset R \\ \ell(Q) = 2^{-r}\ell(R)}} \langle f \rangle_R D_Q a.$$

Prove that given M, γ and a large enough r we have that

$$\|\Pi_a f\|_{L^2(\mu)} \le C \|a\|_{\mathrm{BMO}^2_M(\mu)} \|f\|_{L^2(\mu)}, \qquad f \in L^2(\mu).$$

(5) Let μ be a measure of order m in $\mathbb{R}^n$ and $1 \le p < \infty$. Let $Q \subset \mathbb{R}^n$ be a cube with t-small boundary. Prove that

$$\int_Q \left(\int_{2Q \setminus Q} \frac{d\mu(y)}{|x-y|^m} \right)^p d\mu(x) \le C t \mu(2Q).$$