HARMONIC ANALYSIS AND SQUARE FUNCTIONS: EXERCISE SET 1

You can pass the course by completing the exercises and returning your written solutions. A coherent solution grants 2 points. If there are some mistakes you can still get 1 point. The grades will be assigned as follows:

$$\begin{aligned} 80\% \text{ of total points } &= 5; \\ 73\% \text{ of total points } &= 4; \\ 66\% \text{ of total points } &= 3; \\ 59\% \text{ of total points } &= 2; \\ 50\% \text{ of total points } &= 1. \end{aligned}$$

Return your written solutions preferably directly to Emil Vuorinen (office C435, emil.vuorinen@helsinki.fi). You can also return them during the lectures. **Deadline for the first set is Friday, November** 6**.**

In these exercises we work in $\mathbb{R}^n$, $m \in (0, n]$, (s_t) is an m–LP–family and θ_t, V etc. are like in the lecture notes.

(1) Suppose μ is a measure which is either finite or of order m. Show that for $f \in \bigcup_{p\in[1,\infty]} L^p(\mu)$ and $x \in \mathbb{R}^n$ the integral

$$\theta_t^\mu f(x) = \int s_t(x, y) f(y)\, d\mu(y)$$

is absolutely convergent.

(2) Suppose μ is of order m. Show that $|\theta_t^\mu f(x)| \le CM_\mu f(x)$ for every $t > 0$ and $x \in \mathbb{R}^n$, uniformly in t. Conclude that $\|\theta_t^\mu\|_{L^2(\mu)\to L^2(\mu)}$ is bounded uniformly in $t > 0$. Conclude also that one can for every $i \in \mathbb{N}$ define an m–LP-family $(s_{t,i})_{t>0}$ with kernel constants bounded by those of $(s_t)_{t>0}$, and the related square functions $V_{\mu,i}$ so that $\|V_{\mu,i}\|_{L^2(\mu)\to L^2(\mu)} \le C(i) < \infty$, $V_{\mu,i} f \le V_{\mu,i+1} f \le V_\mu f$, $V_\mu f(x) = \lim_{i\to\infty} V_{\mu,i} f(x)$ and $\|V_\mu f\|_{L^2(\mu)} = \lim_{i\to\infty} \|V_{\mu,i} f\|_{L^2(\mu)}$.

(3) Let $M(\mathbb{R}^n)$ denote the vector-space of all complex measures ν defined in $\mathrm{Bor}(\mathbb{R}^n)$. Show that $M(\mathbb{R}^n)$ is a Banach space when equipped with the norm $\|\nu\| := |\nu|(\mathbb{R}^n)$.

(4) Let μ be a measure of order m and $\beta > \alpha^m$. Show that for every $x \in \operatorname{spt}\mu$ and $c > 0$ there exist some (α, β)-doubling cube Q centred at x with $\ell(Q) \ge c$.

(5) Let μ be a measure of order m. Define the conical square function

$$S_\mu f(x) = \Big(\iint_{\Gamma(x)} |\theta_t^\mu f(y)|^2 \,\frac{d\mu(y)\, dt}{t^{m+1}} \Big)^{1/2},$$

where $\Gamma(x)$ denotes the cone

$$\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ \colon\ |x-y| < t\}.$$

Find an m–LP-family $(\tilde{s}_t)_{t>0}$ so that the related vertical square function $\tilde{V}_\mu$ satisfies

$$\|S_\mu f\|_{L^2(\mu)} = \|\tilde{V}_\mu f\|_{L^2(\mu)}.$$

(6) Let μ be a measure of order m and consider the conical square function S_μ from above. Let us consider an arbitrary fixed ball B centred at c_B. Prove that we have for every $x \in B$ the uniform pointwise bound

$$|S_\mu(1_{\mathbb{R}^n \setminus 10B})(x) - S_\mu(1_{\mathbb{R}^n \setminus 10B})(c_B)| \le C,$$

where $C < \infty$ is independent of x and B.

Prove that the same holds with S_μ replaced by V_μ if one assumes that $(s_t)_{t>0}$ is an *x-continuous* m–LP–family.