EXERCISE SET 2, SUGGESTIONS FOR SOLUTIONS

EMIL VUORINEN

Any comments about the exercises or solutions are warmly welcomed!

Exercise 1. Here we have an arbitrary Radon measure μ and a cube in $\mathbb{R}^n$. The claim is that for any $s > 1$ there holds

$$\int_Q |f| d\mu \leq \frac{s}{s-1} \mu(Q)^{1-\frac{1}{s}} \|f\|_{L^{s,\infty}(\mu)}.$$

The distribution formula relates the integral of f over Q to the weak type norms. Indeed, for any $A > 0$ we have

$$\begin{aligned}
\int_Q |f| d\mu &= \int_0^\infty \mu(Q \cap \{|f| > \lambda\}) d\lambda \\
&\leq \int_0^A \mu(Q) d\lambda + \int_A^\infty \|f\|^s_{L^{s,\infty}(\mu)} \lambda^{-s} d\lambda \\
&= A\mu(Q) + \frac{1}{s-1} \frac{\|f\|^s_{L^{s,\infty}(\mu)}}{A^{s-1}}.
\end{aligned}$$

The minimum of the last quantity occurs when $A = \frac{\|f\|_{L^{s,\infty}(\mu)}}{\mu(Q)^{\frac{1}{s}}}$, and substituting this into above gives the claim.

Exercise 2. Again μ is a Radon measure on $\mathbb{R}^n$, and we consider the maximal function

$$M_{\mu,p} f(x) := \big(M_\mu(|f|^p)(x)\big)^{\frac{1}{p}}, \qquad p \in (1, \infty),$$

where M_μ is the usual centred Hardy-Littlewood maximal function.

Fix some $p \in (1,\infty)$. For any function g and any $s, t > 0$ it holds that $\| |g|^t \|^s_{L^{s,\infty}(\mu)} = \|g\|^{ts}_{L^{ts,\infty}(\mu)}$. Combining this with the fact that the maximal function is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$, we have by Exercise 1 that

$$\begin{aligned}
\frac{1}{\mu(Q)} \int_Q M_{\mu,p} f(x) d\mu(x) &\lesssim \frac{1}{\mu(Q)^{\frac{1}{p}}} \|M_{\mu,p} f\|_{L^{p,\infty}(\mu)} \\
&= \frac{1}{\mu(Q)^{\frac{1}{p}}} \|M_\mu(|f|^p)\|^{\frac{1}{p}}_{L^{1,\infty}(\mu)} \lesssim \frac{1}{\mu(Q)^{\frac{1}{p}}} \| |f|^p \|^{\frac{1}{p}}_{L^1(\mu)},
\end{aligned}$$

which is what we wanted to prove.

Exercise 3. Let μ be a measure of order m in $\mathbb{R}^n$. Assume $Q \subset \mathbb{R}^n$ is a cube and let R be the smallest $(6, 6^{m+1})$-doubling cube of the form $6^k Q,\ k = 0, 1, 2, 3, \ldots$ Claim is that

$$\int_{R \setminus Q} \frac{d\mu(x)}{|x - c_Q|^m} \lesssim 1.$$

Write $R = 6^N Q$ for some N. Because $6^{m+1} > 6^m$ we know by Exercise 4 in the previous set that such a cube R exists. The point is that none of the cubes $Q, 6Q, 6^2 Q, \ldots, 6^{N-1} Q$ is $(6, 6^{m+1})$-doubling, so the measure of the cubes $6^k Q$ grows fast enough that the integral over $6^N Q \setminus 6^{N-1} Q$ dominates the integral over $6^{N-1} Q \setminus Q$. And the integral over $6^N Q \setminus 6^{N-1} Q$ is bounded by some absolute constant, because μ is of order m.

Indeed, for any $k = 0, 1, 2 \ldots, N$, we have $\mu(6^k Q) \le 6^{-(m+1)(N-k)} \mu(R)$. Using this and dividing the integration area into annuli $6Q \setminus Q, 6^2 Q \setminus 6Q, \ldots$, we get

$$\begin{aligned}
\int_{R \setminus Q} \frac{d\mu(x)}{|x - c_Q|^m} &\lesssim \sum_{k=1}^{N} \frac{\mu(6^k Q)}{(6^k l(Q))^m} \\
&\lesssim \sum_{k=0}^{N} \frac{6^{-(m+1)(N-k)} \mu(R)}{(6^k l(Q))^m} = \sum_{k=0}^{N} 6^k 6^{-(m+1)N} \frac{\mu(R)}{l(Q)^m} \\
&\sim \frac{\mu(R)}{(6^N l(Q))^m} \lesssim 1,
\end{aligned}$$

since μ is of order m.

Exercise 4. We prove a claim that was made in the lecture notes during considerations related to weak $(1, 1)$ boundedness of square functions. With the same notation as there we show that for all $x \in \mathbb{R}^n \setminus 2Q_i$ we have

$$V_\nu w_i(x) \lesssim \frac{|\nu|(Q_i)}{|x - c_{Q_i}|^m}.$$

Fix some $x \in \mathbb{R}^n \setminus 2Q_i$. Then for all $y \in Q_i$ we have $|x - y| \sim |x - c_{Q_i}|$ and thus

$$|\theta_t(w_i \nu)| \lesssim \int_{Q_i} \frac{t^\alpha w_i(y) d|\nu|(y)}{(t + |x - y|)^{m+\alpha}} \sim \frac{t^\alpha |\nu|(Q_i)}{(t + |x - c_{Q_i}|)^{m+\alpha}},$$

where we used also that $w_i(y) \sim 1$ for all $y \in Q_i$. Hence

$$\begin{aligned}
V_\nu w_i(x)^2 &\lesssim \int_0^\infty \frac{t^{2\alpha} |\nu|(Q_i)^2}{(t + |x - c_{Q_i}|)^{2(m+\alpha)}} \frac{dt}{t} \\
&\le \int_0^{|x - c_{Q_i}|} t^{2\alpha - 1} \frac{|\nu|(Q_i)^2}{|x - c_{Q_i}|^{2(m+\alpha)}} dt + \int_{|x - c_{Q_i}|}^\infty \frac{|\nu|(Q_i)^2}{t^{2m+1}} dt \\
&\sim \frac{|\nu|(Q_i)^2}{|x - c_{Q_i}|^{2m}}.
\end{aligned}$$

Exercise 5. We have two Radon measure μ and σ and the standard dyadic lattice $\mathcal{D}_0$ on $\mathbb{R}^n$. Assume that $A \subset \mathbb{R}^n$ is a Borel set such that for some $\lambda \ge 0$ we have

$$\underline{D}(\sigma, \mu, x) := \liminf_{k \to \infty} \frac{\sigma(R_k(x))}{\mu(R_k(x))} \le \lambda \tag{0.1}$$

for all $x \in A$, where $R_k(x)$ is the unique cube in $\mathcal{D}_0^k$ that contains x. Then the claim is that $\sigma(A) \le \lambda\mu(A)$.

We can clearly assume that $\mu(A) < \infty$. Let $\varepsilon > 0$ be arbitrary. Because μ is a Radon measure there exists an open set $U \supset A$ such that $\mu(U) \le \mu(A) + \varepsilon$. Because of (0.1) and the fact that U is open, for every $x \in A$ there exists a cube $R \in \mathcal{D}_0$ such that $l(R) \le 1$, $R \subset U$ and

$$\sigma(R) \le \lambda(1+\varepsilon)\mu(R).$$

So if we consider the collection

$$\mathscr{D} := \{R \in \mathcal{D}_0 : R \subset U, l(R) \le 1, \sigma(R) \le \lambda(1+\varepsilon)\mu(R)\}$$

we see that $A \subset \bigcup \mathscr{D}$.

Let now $\tilde{\mathscr{D}}$ be the collection of maximal cubes in $\mathscr{D}$. This means that if $Q \in \tilde{\mathscr{D}}$, then there does not exist a cube $R \in \mathscr{D}$ so that $Q \subsetneq R$. We put the requirement "$l(R) \le 1$" in the definition of $\mathscr{D}$ to be able to choose the maximal cubes. Because every cube $Q \in \mathscr{D}$ is contained in some maximal cube, we clearly have $\bigcup \tilde{\mathscr{D}} = \bigcup \mathscr{D}$.

Next we show that the maximal cubes are pairwise disjoint. This is an important fact that appears in many places and the (very simple) proof illustrates the special properties of dyadic cubes. Suppose $Q, R \in \tilde{\mathscr{D}}$. Because Q and R are dyadic cubes one of the following must hold: $Q \cap R = \emptyset, Q \subset R$ or $R \subset Q$. If $Q \subset R$ or $R \subset Q$, then the maximality of the cubes implies that $Q = R$. Thus either $Q \cap R = \emptyset$ or $Q = R$.

Putting the above pieces together we get

$$\begin{aligned}\sigma(A) &\le \sigma\Big(\bigcup \mathscr{D}\Big) = \sigma\Big(\bigcup \tilde{\mathscr{D}}\Big) = \sum_{R \in \tilde{\mathscr{D}}} \sigma(R) \le \lambda(1+\varepsilon) \sum_{R \in \tilde{\mathscr{D}}} \mu(R) \\ &= \lambda(1+\varepsilon)\mu\Big(\bigcup \tilde{\mathscr{D}}\Big) \le \lambda(1+\varepsilon)\mu(U) \le \lambda(1+\varepsilon)(\mu(A)+\varepsilon).\end{aligned}$$

Since this holds for all $\varepsilon > 0$, the claim is proved.

Exercise 6. Again μ is a Radon measure and $\mathcal{D}_0$ is the standard collection of dyadic cubes in $\mathbb{R}^n$. Suppose that for every $Q \in \mathcal{D}_0$ we have a measurable function A_Q with $\operatorname{spt} A_Q \subset Q$. We define a dyadic square function operator for functions $f \in L^1_{loc}(\mu)$ by

$$Af(x) := \Big(\sum_{Q \in \mathcal{D}_0} |\langle f \rangle_Q^\mu|^2 |A_Q(x)|^2\Big)^{\frac{1}{2}}, \ \ x \in \mathbb{R}^n.$$

The claim is that for $p \in (1, 2]$ there holds

$$\|Af\|_{L^p(\mu)} \lesssim \operatorname{Car}_p((A_Q)_{Q \in \mathcal{D}_0}) \|f\|_{L^p(\mu)}, \tag{0.2}$$

where

$$\operatorname{Car}_p((A_Q)_{Q \in \mathcal{D}_0}) := \sup_{R \in \mathcal{D}_0} \Big(\frac{1}{\mu(R)} \int_R \Big[\sum_{\substack{Q \in \mathcal{D}_0 \\ Q \subset R}} |A_Q(x)|^2\Big]^{\frac{p}{2}} d\mu(x)\Big)^{\frac{1}{p}}.$$

To prove (0.2), it is enough by monotone convergence to take an arbitrary finite subcollection $\widetilde{\mathcal{D}} \subset \mathcal{D}_0$ and prove the bound for the operator

$$\widetilde{A}f(x) := \Big(\sum_{Q \in \widetilde{\mathcal{D}}} |\langle f \rangle_Q^\mu|^2 |A_Q(x)|^2\Big)^{\frac{1}{2}}, \ \ x \in \mathbb{R}^n.$$

Define for every $j \in \mathbb{Z}$ the collection

$$\mathscr{D}_j := \{Q \in \widetilde{\mathcal{D}} : 2^j \le |\langle f \rangle_Q^\mu| < 2^{j+1}\},$$

and denote by $\mathscr{D}_j^*$ the collection of maximal cubes in $\mathscr{D}_j$. These maximal cubes exist because the collection $\mathscr{D}_j$ is finite.

Then, since $\frac{p}{2} \le 1$, we have [1]

$$\begin{aligned}
\|\widetilde{A} f\|_{L^p(\mu)}^p &= \int \Big(\sum_{Q \in \widetilde{\mathcal{D}}} |\langle f \rangle_Q^\mu|^2 |A_Q(x)|^2 \Big)^{\frac{p}{2}} d\mu(x) \\
&\sim \int \Big(\sum_{j \in \mathbb{Z}} 2^{2j} \sum_{Q \in \mathscr{D}_j} |A_Q(x)|^2 \Big)^{\frac{p}{2}} d\mu(x) \le \sum_{j \in \mathbb{Z}} 2^{jp} \int \Big(\sum_{Q \in \mathscr{D}_j} |A_Q(x)|^2 \Big)^{\frac{p}{2}} d\mu(x).
\end{aligned} \tag{0.3}$$

Then, organizing the sums inside the integrals under the maximal cubes (which are disjoint) we get

$$\begin{aligned}
\int \Big(\sum_{Q \in \mathscr{D}_j} |A_Q(x)|^2 \Big)^{\frac{p}{2}} d\mu(x) &= \int \Big(\sum_{R \in \mathscr{D}_j^*} \sum_{\substack{Q \in \mathscr{D}_j \\ Q \subset R}} |A_Q(x)|^2 \Big)^{\frac{p}{2}} d\mu(x) \\
&= \sum_{R \in \mathscr{D}_j^*} \int \Big(\sum_{\substack{Q \in \mathscr{D}_j \\ Q \subset R}} |A_Q(x)|^2 \Big)^{\frac{p}{2}} d\mu(x) \le \sum_{R \in \mathscr{D}_j^*} \mathrm{Car}_p((A_Q)_{Q \in \mathcal{D}_0})^p \mu(R) \\
&= \mathrm{Car}_p((A_Q)_{Q \in \mathcal{D}_0})^p \mu(\bigcup \mathscr{D}_j)
\end{aligned} \tag{0.4}$$

Note that $\bigcup \mathscr{D}_j \subset \{x \in \mathbb{R}^n : M_\mu^d f(x) > 2^j\}$, where M_μ^d is the dyadic maximal function

$$M_\mu^d f(x) := \sup_{x \in Q \in \mathcal{D}_0} \frac{1}{\mu(Q)} \int_Q |f| d\mu.$$

The maximal operator M_μ^d is bounded in $L^p(\mu)$ and hence, using (0.4) in (0.3) leads to

$$\begin{aligned}
\|\widetilde{A} f\|_{L^p(\mu)}^p &\lesssim \mathrm{Car}_p((A_Q)_{Q \in \mathcal{D}_0})^p \sum_{j \in \mathbb{Z}} 2^{jp} \mu(\{x \in \mathbb{R}^n : M_\mu^d f(x) > 2^j\}) \\
&\lesssim \mathrm{Car}_p((A_Q)_{Q \in \mathcal{D}_0})^p \sum_{j \in \mathbb{Z}} \int_{2^{(j-1)p}}^{2^{jp}} \lambda^{p-1} \mu(\{x \in \mathbb{R}^n : M_\mu^d f(x) > \lambda\}) d\lambda \\
&\sim \mathrm{Car}_p((A_Q)_{Q \in \mathcal{D}_0})^p \|M_\mu^d f\|_{L^p(\mu)}^p \lesssim \mathrm{Car}_p((A_Q)_{Q \in \mathcal{D}_0})^p \|f\|_{L^p(\mu)}^p.
\end{aligned}$$

Remark 1. Note that the constant $\mathrm{Car}_p((A_Q)_{Q \in \mathcal{D}_0})$ is also a kind of testing constant for the operator A. Indeed, if you define for every $R \in \mathcal{D}_0$ the localized version of the square function by

$$A^R f(x) := \Big(\sum_{\substack{Q \in \mathcal{D}_0 \\ Q \subset R}} |\langle f \rangle_Q^\mu|^2 |A_Q(x)|^2 \Big)^{\frac{1}{2}},$$

[1]Note that we used the fact that if $0 < p < q < \infty$, then for any sequence $(x_n)_{n=1}^\infty \subset \mathbb{R}$ we have $\Big(\sum_n |x|^p \Big)^{\frac{1}{p}} \ge \Big(\sum_n |x|^q \Big)^{\frac{1}{q}}$.

then

$$\operatorname{Car}_p((A_Q)_{Q\in\mathcal{D}_0}) = \sup_{R\in\mathcal{D}_0} \frac{\|A^R 1_R\|_{L^p(\mu)}}{\|1_R\|_{L^p(\mu)}}.$$

In particular, if for some $p \in (1,2]$ you know that there exists some constant C such that $\|A1_R\|_{L^p(\mu)} \leq C\|1_R\|_{L^p(\mu)}$ for all $R \in \mathcal{D}_0$, then you know that A is bounded in $L^p(\mu)$. This means that to get the boundedness of A it is enough to test it with indicators of dyadic cubes!

Exercise 7. In this Exercise we have a Radon measure μ in $\mathbb{R}^n$ and in every cube $Q \in \mathcal{D}_0$ a function φ_Q such that

- $\operatorname{supp}\varphi_Q \subset Q$.
- φ_Q is constant on the children of Q.
- $\|\varphi_Q\|_{L^\infty(\mu)} \leq 1$.

For every $P \in \mathcal{D}_0$ we define the function

$$\Phi_P := \sup_{\varepsilon>0} \Big| \sum_{\substack{Q\subset P \\ l(Q)>\varepsilon}} \varphi_Q \Big|.$$

Then assume for every $P \in \mathcal{D}_0$ that

$$\mu(\{\Phi_P > 1\}) \leq \frac{1}{2}\mu(P),$$

and the claim is that for every $t > 1$ we have

$$\mu(\{\Phi_P > t\}) \leq 2^{-(t-1)/2}\mu(P). \tag{0.5}$$

Proof. Because each φ_Q is constant on the children of Q, it can be written as

$$\varphi_Q = \sum_{Q'\in\operatorname{ch}(Q)} c_{Q'} 1_{Q'}$$

for some constants $c_{Q'}$ such that $|c_{Q'}| \leq 1$. Thus also the function Φ_P can be written as

$$\Phi_P = \sup_{\varepsilon>0} \Big| \sum_{\substack{Q\subsetneq P \\ l(Q)>\varepsilon}} c_Q 1_Q \Big|.$$

For any $Q_0 \in \mathcal{D}_0$ let $\mathcal{F}(Q_0)$ denote the maximal cubes (if they exist) $Q' \in \mathcal{D}_0, Q' \subsetneq Q_0$, such that

$$\Big| \sum_{Q\colon\, Q'\subset Q\subsetneq Q_0} c_Q \Big| > 1.$$

If $Q' \in \mathcal{F}(Q_0)$, then clearly $\Phi_{Q_0}(x) > 1$ for all $y \in Q'$. On the other hand if $\Phi_{Q_0}(x) > 1$, then there clearly exists a $Q' \in \mathcal{F}(Q_0)$ such that $x \in Q'$. Hence

$$\{\Phi_{Q_0} > 1\} = \bigcup_{Q'\in\mathcal{F}(Q_0)} Q', \tag{0.6}$$

and the assumptions imply that $\mu(\bigcup_{Q'\in\mathcal{F}(Q_0)} Q') = \sum_{Q'\in\mathcal{F}(Q_0)} \mu(Q') \leq \frac{1}{2}\mu(Q_0)$.

To prove (0.5), fix some cube $P \in \mathcal{D}_0$ and $t > 1$. Looking at the claim we see that we can actually assume $t > 3$. Let $m \in \mathbb{Z}$ be largest so that $t - 2m > 1$. Suppose now

$x \in \{\Phi_P > t\}$. Then by (0.6) we now that there exists a cube $Q' \in \mathcal{F}(P)$ such that $x \in Q'$. Because $\Phi_P(x) > t$ there exists some $S \in \mathcal{D}_0, x \in S$, such that

$$\Big| \sum_{Q\colon S \subset Q \subsetneq Q_0} c_Q \Big| > t. \tag{0.7}$$

The maximality in the definition of $Q' \in \mathcal{F}(P)$ implies that $S \subset Q'$.

But on the other hand, again by the maximality $Q' \in \mathcal{F}(P)$ and the fact that $|c_{Q'}| \leq 1$, we have

$$\begin{aligned}
\Big| \sum_{Q\colon S \subset Q \subsetneq Q_0} c_Q \Big| &\leq \Big| \sum_{Q\colon S \subset Q \subsetneq Q'} c_Q \Big| + |c_{Q'}| + \Big| \sum_{Q\colon Q' \subsetneq Q \subsetneq P} c_Q \Big| \\
&\leq \Big| \sum_{Q\colon S \subset Q \subsetneq Q'} c_Q \Big| + 2,
\end{aligned}$$

which, combined with (0.7), implies that

$$\Phi_{Q'}(x) \geq \Big| \sum_{Q\colon S \subset Q \subsetneq Q'} c_Q \Big| > t - 2.$$

Collecting the above considerations we have shown that

$$\{\Phi_P > t\} \subset \bigcup_{Q' \in \mathcal{F}(P)} \{x \in Q' \colon \Phi_{Q'}(x) > t - 2\}.$$

Then, beginning with the cubes $Q' \in \mathcal{F}(P)$ we can continue this process inductively to get

$$\begin{aligned}
\{\Phi_P > t\} &\subset \bigcup_{Q_i^1 \in \mathcal{F}(P)} \{x \in Q_i^1 \colon \Phi_{Q_i^1}(x) > t - 2\} \\
&\subset \bigcup_{Q_i^1 \in \mathcal{F}(P)} \bigcup_{Q_j^2 \in \mathcal{F}(Q_i^1)} \{x \in Q_j^2 \colon \Phi_{Q_j^2}(x) > t - 4\} \\
&\subset \bigcup_{Q_i^1 \in \mathcal{F}(P)} \cdots \bigcup_{Q_k^m \in \mathcal{F}(Q_j^{m-1})} \{x \in Q_k^m \colon \Phi_{Q_k^m}(x) > t - 2m\}.
\end{aligned}$$

Remember that $3 \geq t - 2m > 1$. Since the collections $\mathcal{F}(Q)$ consist of pairwise disjoint cubes we can estimate

$$\begin{aligned}
\mu(\{\Phi_P > t\}) &\leq \sum_{Q_i^1 \in \mathcal{F}(P)} \cdots \sum_{Q_k^m \in \mathcal{F}(Q_j^{m-1})} \mu(\{x \in Q_k^m \colon \Phi_{Q_k^m}(x) > t - 2m\}) \\
&\leq \sum_{Q_i^1 \in \mathcal{F}(P)} \cdots \sum_{Q_k^m \in \mathcal{F}(Q_j^{m-1})} \frac{1}{2} \mu(Q_k^m) \\
&\leq \sum_{Q_i^1 \in \mathcal{F}(P)} \cdots \sum_{Q_j^{m-1} \in \mathcal{F}(Q_l^{m-2})} \frac{1}{4} \mu(Q_k^{m-1}) \\
&\leq \frac{1}{2^{m+1}} \mu(P) \leq 2^{-\frac{t-1}{2}} \mu(P).
\end{aligned}$$

This concludes the proof. □

Exercise 8. In the situation of the previous Exercise we want to show for every $P \in \mathcal{D}_0$ that

$$\int_P |\Phi_P|^p d\mu \leq C(p)\mu(P), \quad p \in (0, \infty).$$

The idea is that (0.5) implies that as $t \to \infty$ the measure of $\{\Phi_P > t\}$ goes to zero very fast, and a way to use this information is some estimate for the integral using the size of the level sets $\{\Phi_P > t\}$. Indeed, using for example the distribution formula, estimate as

$$\begin{aligned} \int_P |\Phi_P|^p d\mu &= \int_0^\infty p\lambda^{p-1}\mu(\{\Phi_P > \lambda\})d\lambda \\ &\leq \int_0^1 p\lambda^{p-1}\mu(P)d\lambda + \int_1^\infty p\lambda^{p-1}2^{-\frac{\lambda-1}{2}}\mu(P)d\lambda, \end{aligned}$$

and what is left to do is to show that

$$\int_1^\infty \frac{\lambda^{p-1}}{2^{\frac{\lambda-1}{2}}} < \infty.$$

Let $m \in \mathbb{Z}$ be the smallest integer $m > p$. Write $2^{\frac{\lambda-1}{2}} = e^{\frac{\lambda-1}{2}\ln 2} = e^{-\frac{1}{2}\ln 2}e^{\frac{\lambda}{2}\ln 2} = Ce^{c\lambda}$. Then

$$\int_1^\infty \frac{\lambda^{p-1}}{2^{\frac{\lambda-1}{2}}} = \int_1^\infty \frac{\lambda^{p-1}}{Ce^{c\lambda}}d\lambda \leq \int_1^\infty \frac{\lambda^{p-1}}{C\frac{(c\lambda)^m}{m!}}d\lambda < \infty,$$

since $m - p + 1 > 1$.