Matematiikan ja tilastotieteen laitos
Fourier analyysi
Review problems

(These are extra and voluntary !!)

Aleksis Koski will be on call in his office A415 on Monday 19th October at 12-14, to give help/answers/advice on these problems.

1. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence in $L^1[-\pi,\pi]$ and $f \in L^1[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi} |f_k(x) - f(x)| dx \xrightarrow{k\to\infty} 0.$$

Show that $\hat{f}_k(n) \to \hat{f}(n)$ uniformly with respect to n as $k \to \infty$.

2. Suppose $f(x)$ is a 2π-periodic function such that $f(x) = 0$ for $-\pi \le x \le 0$ and $f(x) = \sin(1/x)$ for $0 < x \le \pi$. Show that

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| = \infty,$$

i.e. the Fourier series of the function f does not converge absolutely.

3. Let $f \in L^1(\mathbb{R})$ be a 2π-periodic function satisfying $f(x+\pi) = -f(x)$ for every $x \in [-\pi,\pi]$.

Show that the even Fourier coefficients of f are zero, i.e. prove that $\hat{f}(2n) = 0$ for every $n \in \mathbb{Z}$.

4. Let $N \in \mathbb{N}$. Show that there exists a function $f \in L^1[-\pi,\pi]$ such that $F_N * f = 0$.

5. If a function $f \in L^1(-\pi,\pi)$ has a derivative at a point $x_0 \in [-\pi,\pi]$, show that then the Fourier series of f converges at x_0, and has value $f(x_0)$.

6. Suppose $f \in L^2[-\pi,\pi]$ is a function for which

$$\sum_{n=-\infty}^{\infty} n^2 |\hat{f}(n)|^2 < \infty.$$

Use the Riesz-Fisher theorem to show that there is a function $g \in L^2[-\pi,\pi]$ such that $\hat{g}(n) = in\hat{f}(n)$, $n \in \mathbb{Z}$, and that g satisfies

$$(*) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{\phi'(x)}dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\overline{\phi(x)}dx$$

for every 2π-periodic function $\phi \in C^\infty(\mathbb{R})$.

[Note: a function g satisfying $(*)$ for all $\phi \in C^\infty_\#(-\pi,\pi)$ is called a *weak derivative* of f.]

7. Determine the Fourier coefficients of the function $f(x) = |x|,\ x \in [-\pi,\pi]$, and use these and f to show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

8. Suppose $\{A_n(x)\}_{n\in\mathbb{N}}$ and $\{B_n(x)\}_{n\in\mathbb{N}}$ are families of 2π-periodic good kernels. For which numbers $\alpha, \beta \in \mathbb{C}$ is

$$\{\alpha A_n(x) + \beta B_n(x)\}_{n\in\mathbb{N}}$$

a family of good kernels ? Justify your answer.