Department of Mathematics and Statistics
Fourier analysis
Exercise 9
Nov. 23, 2015

1. i) If $p_N(f) = \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N\, |\partial^\alpha f(x)|$ and

$$\rho(f,g) := \sum_{N=0}^{\infty} 2^{-N}\, \frac{p_N(f-g)}{1+p_N(f-g)}, \qquad f,g \in \mathcal{S}(\mathbb{R}^d),$$

show that $\rho(f,g)$ defines a metric in the space $\mathcal{S}(\mathbb{R}^d)$.

ii) If $f, f_n \in \mathcal{S}(\mathbb{R}^d)$, $n \in \mathbb{N}$, show that $\rho(f_n, f) \to 0$ as $n \to \infty$ if and only if for every $N \in \mathbb{N}$, $p_N(f - f_n) \to 0$ as $n \to \infty$.

iii) Show that Schwartz space $\mathcal{S}(\mathbb{R}^d)$ equipped with the metric $\rho(f,g)$ is complete.

2. Show that a linear $T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is continuous

$\Leftrightarrow$ for some $N \in \mathbb{N}$ and $C < \infty$ we have:

$$|T(f)| \;\le\; C \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N\, |\partial^\alpha f(x)|, \qquad \forall\, f \in \mathcal{S}(\mathbb{R}^d).$$

3. (i) Given $a \in \mathbb{R}^d$ and $g \in \mathcal{S}(\mathbb{R}^d)$, let

$$T_1(g) \;=\; \int_{-1}^{1} g(ta)\, dt.$$

(ii) Given $g \in \mathcal{S}(\mathbb{R})$, let

$$T_2(g) \;=\; \sum_{n\in\mathbb{Z}} g'(n).$$

Show that the linear maps $T_1 : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ and $T_2 : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ are both continuous.

4. For $0 < r < \infty$, let $g_r(x) = e^{-r|x|^2}$, $x \in R^d$.
Determine the convolution $g_{r_1} * g_{r_2}$.

[Hint: Calculations might be easier on the Fourier side.]

5. (Young's convolution inequality) If

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}, \qquad 1 \le p, q, r \le \infty,$$

and if $f \in L^p(\mathbb{R}^d)$, $g \in L^r(\mathbb{R}^d)$, show that $f * g \in L^q(\mathbb{R}^d)$ and

$$\|f * g\|_{L^q(\mathbb{R}^d)} \ \le \ \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^r(\mathbb{R}^d)}.$$

[Hint: Fubini and elementary estimates give $\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$ with $\|f * g\|_{L^\infty} \le \|g\|_{L^\infty}\|f\|_{L^1}$. Use these with Riesz-Thorin for $\|f * g\|_{L^r} \le \|g\|_{L^r}\|f\|_{L^1}$.
Use Hölder's inequality for $\|f * g\|_{L^\infty} \le \|g\|_{L^r}\|f\|_{L^{r'}}$, where $r' = \frac{r}{r-1}$, and apply Riesz-Thorin again.]