Department of Mathematics and Statistics
Fourier analysis
Exercise 8
Nov. 16, 2015

1. If $f \in \mathcal{S}(\mathbb{R}^d)$, show that $(1+|x|^2)^{-1} f(x) \in \mathcal{S}(\mathbb{R}^d)$.

2. Let $f \in C^\infty(\mathbb{R}^d)$ be defined by $f(x) = \sin(e^{|x|^2})e^{-|x|^2}$. Is $f \in \mathcal{S}(\mathbb{R}^d)$?

3. (i) Using the identity $\int_{\mathbb{R}^d} f(x)\widehat{g}(x)dx = \int_{\mathbb{R}^d} \widehat{f}(x)g(x)dx$
(c.f. Proposition 9.7) and properties of Schwarz functions, show that

$$(2\pi)^d \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi \tag{1}$$

whenever $f, g \in \mathcal{S}(\mathbb{R}^d)$.
(ii) Using the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, deduce from (i) that Parseval's identity (1) holds for every $f, g \in L^2(\mathbb{R}^d)$.

4. If $h \in \mathcal{S}(\mathbb{R}^d)$, show that the differential equation

$$\Delta f - f = h, \qquad \Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_d}\right)^2,$$

has always a solution $f \in \mathcal{S}(\mathbb{R}^d)$.

[Hint: Determine $\mathcal{F}(\Delta f)$ and take Fourier transform of the equation. The first problem might be useful.]

5. Prove the Heisenberg uncertainty principle:

If $f \in \mathcal{S}(\mathbb{R})$ and $\int_{-\infty}^{\infty} |f(x)|^2\, dx = 1$, then

$$\frac{\pi}{2} \leq \int_{\mathbb{R}} |x|^2\, |f(x)|^2 dx \int_{\mathbb{R}} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi.$$

[Hint: Express the integral $\int_{-\infty}^{\infty} x \frac{d}{dx}\big[f(x)\overline{f(x)}\,\big]\, dx$ in two different ways.]