Department of Mathematics and Statistics
Fourier analysis
Exercise 7
Nov. 9, 2015

1. Find an example of a function $g \in L^2(\mathbb{R})$, such that
$\widehat{g} \in L^1(\mathbb{R})$ but $g \notin L^1(\mathbb{R})$.

2. Suppose $K_t : \mathbb{R}^d \to \mathbb{C}$ are measurable functions such that, for all $t > 0$ $\int_{\mathbb{R}^d} K_t(x)dx = 1$ and $\int_{\mathbb{R}^d} |K_t(x)|dx \leq C_0$, and that $\forall \delta > 0$,

$$\int_{|x|>\delta} |K_t(x)|dx \to 0 \quad \text{as} \quad t \to 0$$

(in this case we say $\{K_t\}_{t>0}$ is a *family of good kernels in* $\mathbb{R}^d$, cf. notes p. 21).

Let $f \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Show that

(i) If f is continuous at the point $x_0 \in \mathbb{R}^d$, then $(K_t * f)(x_0) \to f(x_0)$ as $t \to 0$.

(ii) For all $\xi \in \mathbb{R}^d$ we have $(\widehat{K_t * f})(\xi) \to \widehat{f}(\xi)$ as $t \to 0$.

3. Suppose $\alpha \in \mathbb{N}^d$ is a multi-index. Show that if $f \in \mathcal{S}(\mathbb{R}^d)$, then
(i) $x^\alpha f(x) \in \mathcal{S}(\mathbb{R}^d)$ and $\partial^\alpha f(x) \in \mathcal{S}(\mathbb{R}^d)$,
(ii) $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$,
(iii) $\widehat{f} \in C^\infty(\mathbb{R}^d)$, and $\partial^\alpha \widehat{f}(\xi) = \Big((-i\,x)^\alpha f(x) \Big)\widehat{\ }(\xi)$
(iv) $(\partial^\alpha f)\widehat{\ }(\xi) = (i\,\xi)^\alpha \widehat{f}(\xi)$.
Above $i^\alpha := i^{|\alpha|}$.

4. (i) If $f(x) = e^{-|x|^2}$, for $x \in \mathbb{R}^d$, show that $f \in \mathcal{S}(\mathbb{R}^d)$.
(ii) If $f \in \mathcal{S}(\mathbb{R}^d)$, show that $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$.

5. Suppose the Fourier transform of a function $f \in L^1(\mathbb{R})$ satisfies the condition

$$|\widehat{f}(\xi)| \leq \frac{C}{(1 + |\xi|)^{1+a}}, \qquad \xi \in \mathbb{R},$$

for some constants $0 < a < 1$ and $C < \infty$. Show that then $f \in Lip_a(\mathbb{R})$, that is,

$$|f(x+h) - f(x)| \leq M\,|h|^a, \qquad x \in \mathbb{R}, \quad h \in \mathbb{R}.$$

[Hint: The formula for the inverse Fourier transform, when $f,\, \widehat{f} \in L^1(\mathbb{R})$, might be useful.]