Department of Mathematics and Statistics
Fourier analysis
Exercise 6
Nov. 2, 2015

1. Show that the periodic heat kernel $H_t(x) := \sum_{n=-\infty}^{\infty} e^{-n^2 t} e^{inx}$ and the Dirichlet heat kernel $K_t(x,y) := \sum_{n=1}^{\infty} e^{-n^2 t} \sin(ny) \sin(nx)$ are connected by the following relation:

$$(H_t * f)(x) = \frac{2}{\pi} \int_0^{\pi} K_t(x,y)\, f(y)\, dy$$

for every *odd* function $f \in L^2(-\pi,\pi)$ and for every $x \in [0,\pi]$.

2. Consider the function in $\mathbb{R}^1$ defined by $f(x) := e^{-k|x|}$ (where $k > 0$). Show that its Fourier transform $\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx$ is given by

$$\widehat{f}(\xi) = \frac{2k}{k^2 + \xi^2}, \qquad \xi \in \mathbb{R}.$$

3. (i) Suppose that the function $f : \mathbb{R}^2 \to \mathbb{C}$ has the form

$$f(x_1, x_2) = f_1(x_1) f_2(x_2), \qquad \forall\, x = (x_1, x_2) \in \mathbb{R}^2,$$

where $f_1, f_2 \in L^1(\mathbb{R}^1)$. Show that then $f \in L^1(\mathbb{R}^2)$ and we have

$$\widehat{f}(\xi_1, \xi_2) = \widehat{f}_1(\xi_1) \widehat{f}(\xi_2) \qquad \forall\, \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

(ii) If $f \in L^1(\mathbb{R}^d)$ and $g(x) = \overline{f(-x)}$, show that $\widehat{g}(\xi) \equiv \overline{\widehat{f}(\xi)}$.

(ii) If $f \in L^1(\mathbb{R}^d)$ and $g(x) = \frac{1}{t^d} f\left(\frac{x}{t}\right)$, $t > 0$, show that $\widehat{g}(\xi) \equiv \widehat{f}(t\,\xi)$.

4. If $f \in L^2(0,\pi)$ and $A_n(f) := \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$, $n \geq 1$, show that

$$\int_0^{\pi} |f(x) - \sum_{n=1}^{N} A_n(f) \sin(nx)|^2\, dx \to 0$$

and that

$$\frac{2}{\pi} \int_0^{\pi} |f(x)|^2\, dx = \sum_{n=1}^{\infty} |A_n(f)|^2.$$

[Hint: Extend f to $[-\pi,\pi]$ by a reflection, i.e. set $f(-x) = -f(x)$, and express in terms of a sine-series the partial Fourier sums $S_N f$ of the extension.]

5. (i) Determine the Fourier transform of the characteristic function $\chi_{[-a,a]}$ of the interval $[-a, a]$.

(that is, $\chi_{[-a,a]}(x) = 1$, if $|x| \leq a$ and $\chi_{[-a,a]}(x) = 0$ if $|x| > a$.)

(ii) Show that $\widehat{f} \notin L^1(\mathbb{R})$. Is $\widehat{f} \in L^2(\mathbb{R})$?

(iii) More generally, determine the Fourier transform of the characteristic function of the cube $[-a, a]^d$.