Department of Mathematics and Statistics
Fourier analysis
Exercise 5
Oct. 12, 2015

1. (a) Show that if $f_n \to f$ and $g_n \to g$ in $L^2(-\pi,\pi)$ (i.e. converging in the L^2-norm), then

$$(f_n, g_n)_{L^2} \to (f,g)_{L^2} \qquad \text{as } n \to \infty.$$

(b) Prove the Pythagorean theorem in $L^2(-\pi,\pi)$, that is, show that

$$\|f+g\|_{L^2}^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2 \qquad \text{if} \quad f \perp g \quad \text{and} \quad f,g \in L^2(-\pi,\pi).$$

2. Suppose $f \in C^1_\#(-\pi,\pi)$. Show that the Fourier series of f converges absolutely, i.e. we have $\sum |\widehat{f}(n)| < \infty$.
[Hint: Determine $\widehat{f}(n)$ in terms of $\widehat{f'}(n)$, and recall that the Cauchy-Schwarz inequality holds for the inner product $(a,b)_{\ell^2} = \sum_{k=-\infty}^{\infty} a_k \overline{b_k}$ in the space ℓ^2, see Lecture notes p. 58 (the C-S holds in every inner product space).]

3. Suppose $f \in C^1[0,\pi]$ with $f(0) = 0 = f(\pi)$. Prove Wirtinger's inequality,

$$(*) \qquad \int_0^\pi |f(x)|^2 dx \le \int_0^\pi |f'(x)|^2 dx.$$

Show also that the equality holds in $(*)$ for some functions $f \neq 0$. Can you identify these ?
[Hint: Extend f as an *odd* function to the interval $[-\pi,\pi]$ and represent the functions as a Fourier series.]

4. Determine the Fourier series of the 2π-periodic function f defined by $f(x) = 1$ for $|x| < 1$ and $f(x) = 0$ when $x \in [-\pi,\pi] \setminus [-1,1]$.

Use the series and Plancherel's formula, or Thm. 6.5 in Lecture notes, to calculate the sum $\sum_{n=1}^{\infty} \frac{sin^2(n)}{n^2}$.

5. Suppose the Fourier series of a function $g \in C_\#(-\pi,\pi)$ is a lacunary series of the form

$$\sum_{k=-\infty}^{\infty} a_k e^{i2^{|k|}x}.$$

Show that then the partial Fourier sums are uniformly bounded, i.e. $|S_n g(x)| \le C$ for some constant $C < \infty$ and for all $n \in \mathbb{N}$.