Department of Mathematics and Statistics
Fourier analysis
Exercise 3
Sept. 21, 2015

[*Note*: There will be no exercises on Sept. 28; next exercises Oct.5]

1. Suppose $f(x)$ and $g(x)$ are 2π-periodic functions with $f, g \in L^1[-\pi, \pi]$. Show that if either $f \in C_\#(-\pi, \pi)$ or $g \in C_\#(-\pi, \pi)$, then the convolution $f * g \in C_\#(-\pi, \pi)$.

Prove that the claim holds also when $C_\#(-\pi, \pi)$ is replaced by $C^1_\#(-\pi, \pi)$.

2. a) If a series $\sum_{n=0}^\infty a_n$ converges, show that it converges also in the Cesàro sense.

b) Suppose $f : \mathbb{R} \to \mathbb{C}$ is continuous and 2π-periodic. If the Fourier series of f converges at $x_0 \in [-\pi, \pi]$, that is, if

$$\exists \lim_{N\to\infty} S_N f(x_0) = a \in \mathbb{C},$$

show that the limit must necessarily be the value of f at x_0, i.e. we have $a = f(x_0)$.

3. Show that the Fourier coefficients of the Fejer kernel are given by

$$\widehat{F_N}(k) = \left(1 - \frac{|k|}{N}\right), \qquad \text{when } |k| \le N,$$

while $\widehat{F_N}(k) = 0$ when $|k| \ge N$.

4. a) Show that for every 2π-periodic function $f \in L^1[-\pi, \pi]$ we have

$$\widehat{f}(n) = \frac{1}{4\pi} \int_0^{2\pi} e^{-inx}\big(f(x) - f(x + \pi/n)\big)dx.$$

b) If $f \in C_\#(-\pi, \pi)$ is Hölder-continuous with exponent $\alpha \in (0, 1]$, show that

$$|\widehat{f}(n)| \le C|n|^{-\alpha}, \quad \text{for } |n| \ge 1.$$

5. Prove Corollary 4.8; that is, show that if a 2π-periodic function $f(x)$ is piecewise C^1, then its Fourier series converges at every point, and

$$\lim_{N\to\infty} S_N f(x) = \lim_{t\to 0} \frac{f(x+t) + f(x-t)}{2}, \qquad x \in [-\pi, \pi].$$