Department of Mathematics and Statistics
Fourier analysis
Exercise 2
Sept. 14, 2015

1. Which of the following families $\{K_n\}_{n=1}^{\infty}$ of 2π-periodic functions is a family of *good kernels*, which not:

$$i)\ K_n(x) = 2\pi n \max\{0,\, 1-n|x|\}, \quad ii)\ K_n(x) = \frac{\sin(nx)}{x^2}, \quad iii)\ K_n(x) = (n+1)\left(1 - \frac{|x|}{\pi}\right)^n.$$

The kernels $K_n(x)$ above are all defined on the interval $x \in [-\pi, \pi]$.

2. Let $f(x)$ and $g(x)$ be 2π-periodic functions such that $f,\, g \in L^1[-\pi, \pi]$. Show that the Fourier coefficients of the convolution $f * g$ are given by

$$(\widehat{f * g})(n) = \widehat{f}(n)\,\widehat{g}(n), \quad n \in \mathbb{N}.$$

[Hint: You may use e.g. Fubini's theorem]

3. Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ defined by $f(x) = \cos(x/2)$. Determine the Fourier coefficients of f. Does the corresponding Fourier series converge at every point ? If yes, what is the identity you find when evaluating the series at $x = 0$?

4. Show that the Dirichlet kernels satisfy the following identity,

$$\sum_{k=0}^{N-1} D_k(x) = \left(\frac{\sin(Nx/2)}{\sin(x/2)}\right)^2, \qquad N \in \mathbb{N}.$$

[Hint: Show that $(e^{ix/2} - e^{-ix/2})D_k(x) = e^{ix/2}e^{ikx} - e^{-ix/2}e^{-ikx}$, $k \in \mathbb{N}$, using identities from lectures. Sum this over indices $k = 0, 1, \ldots, N-1$.]

5. From lectures we know that the Fourier coefficients $\widehat{f}(n)$ of a function $f \in C_{\#}^{(k)}$ decay to zero with the speed $|n|^{-k}$ as $n \to \infty$.

Show that a converse result holds in the following sense: If f is continuous on the interval $[-\pi, \pi]$ and for every $k \in \mathbb{N}$ there is a constant $C = C_k$ for which

$$|\widehat{f}(n)| \le C_k(1 + |n|)^{-k}, \quad n \in \mathbb{Z},$$

then $f \in C_{\#}^{\infty}(-\pi, \pi) := \bigcap_{k=1}^{\infty} C_{\#}^{(k)}(-\pi, \pi)$.

[Hint: Show that you can derivate the Fourier series]