Department of Mathematics and Statistics
Fourier analysis
Exercise 12 (Voluntary)
Dec. 14, 2015

1. (i) Show that $< \mathcal{F}^{-1}T, g > \, = \, < T, \mathcal{F}^{-1}g >$ for all $T \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

   (ii) If $T := p.v.\ \frac{1}{i\xi}$, show that $i\xi\, T = 1$, as elements of $\mathcal{S}'(\mathbb{R})$.

2. Define the principal value $p.v.\frac{1}{x^2}$ as a distribution in $\mathcal{S}'(\mathbb{R})$, by setting

$$\langle\, p.v.\frac{1}{x^2}\ , g\rangle := \lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{g(x)-g(0)}{x^2}dx, \qquad \text{when} \quad g \in \mathcal{S}(\mathbb{R}).$$

Show that this is indeed a tempered distribution.

3. If $T = p.v.\frac{1}{x}$ interpreted as a distribution, c.f. Lectures, show that the (distributional) derivative of T is $-p.v.\frac{1}{x^2}$, the distribution defined in Problem 2 multiplied by -1.

What is the Fourier transform of $p.v.\frac{1}{x^2}$?

4. Let $A = \{(x,y) : x > 0, y > 0\} \cup \{(x,y) : x < 0, y < 0\} \subset \mathbb{R}^2$.

   Show that the characteristic function χ_A is a fundamental solution (perusratkaisu) for the differential operator $P_1(\partial) = \frac{1}{2}\partial_1\partial_2$.

5. Let $B = \{(x,y) : y > |x|\} \cup \{(x,y) : y < -|x|\} \subset \mathbb{R}^2$; sketch a picture.

   Show that $\frac{-1}{4}\chi_B$ is a fundamental solution for the wave operator

$$P_2(\partial) = \partial_1^2 - \partial_2^2.$$

[Hint: Determine $P_2(\partial)g$ for the function $g(x,y) = h(x+y, y-x)$ and change variables.]

Additional Review Problems

6. i) Let $f \in L^1(\mathbb{R})$. If $|\xi|\widehat{f}(\xi) \in L^1(\mathbb{R})$, show that

$$f \in C^1(\mathbb{R}).$$

ii) if $f = \chi_{[-1,1]} * \chi_{[-1,1]} * \cdots * \chi_{[-1,1]}$, where f is the convolution of $(n+2)$ characteristic functions, $n \geq 0$, show that

$$f \in C^{(n)}(\mathbb{R}).$$

7. Let $\phi(x) = e^{iax} - e^{ibx}$, $x \in \mathbb{R}$, where $a, b \in \mathbb{R}$ are constants. From the lectures we know how to multiply a tempered distribution by ϕ. Show that

$$T \;:=\; \phi(x)\; p.v.\frac{1}{x} \;\in\; C_0(\mathbb{R}).$$

In other words, show that $T = T_f$ for some function $f \in C_0(\mathbb{R})$. Determine the function f.

8. If $a, b \in \mathbb{R}$ with $0 < b < 1 < a$ and $ab > 1$, recall the Weierstrass functions

$$f(x) \;=\; \sum_{n=1}^{\infty} b^n\, e^{i\,a^n x}, \qquad x \in \mathbb{R}.$$

Then $f \in \mathcal{S}'(\mathbb{R})$ (why). Show that the *distributional* derivate of this function is

$$f' \;=\; \sum_{n=1}^{\infty} i a^n b^n\, e^{i\,a^n x}.$$

where the sum converges in $\mathcal{S}'(\mathbb{R})$.
[Although the function is *nowhere* differentiable, in the classical sense !]