Department of Mathematics and Statistics
Fourier analysis
Exercise 11
Dec. 7, 2015

1. Is the function $f(x) = x\sin(x)$ the Fourier transform of some distribution $T \in \mathcal{S}'(\mathbb{R})$? If it is, determine T.

2. (i) Suppose $A : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear map (we denote by A also its matrix). If $f \in L^1(\mathbb{R}^d)$, define $g(x) = f(Ax)$. Show that

$$\widehat{g}(\xi) = \frac{1}{|\det(A)|}\widehat{f}((A^{-1})^T\xi),$$

where $(A^{-1})^T$ is the transpose of the inverse of A.

(ii) A function $f \in L^1(\mathbb{R}^d)$ is radial if $f(x)$ depends only on $|x|$. Use (i) to show that for a radial function, the Fourier transform is radial.

(iii) Show that the result in (ii) holds also for every radial $f \in L^2(\mathbb{R}^d)$.

3. Show that $f(x) = \log|x| \in \mathcal{S}'(\mathbb{R})$ and that the distributional derivative of f is

$$\frac{d}{dx}(\log|x|) = \mathbf{p.v.}\frac{1}{x}$$

4. Use the Poisson summation formula to prove

$$\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \pi\frac{1+e^{-2\pi}}{1-e^{-2\pi}}$$

[Hint: Recall the Fourier transform of $f(x) = e^{-|x|}$.]

5. (i) If $0 < \gamma < d$, show that the function

$$f_\gamma(x) = \frac{1}{|x|^\gamma}, \qquad x \in \mathbb{R}^d \setminus \{0\},$$

determines a tempered distribution, by writing it as a sum of two functions, one belonging to $L^1(\mathbb{R}^d)$ and the other to $L^p(\mathbb{R}^d)$ for a suitable $p > 1$. If $\gamma > d/2$, show that one can choose $p = 2$.

(ii) Prove that

$$\widehat{f_\gamma}(\xi) = c(d,\gamma)\frac{1}{|x|^{d-\gamma}}$$

for some constant $c(d,\gamma)$.

[Hints: Consider first the case $\gamma > d/2$ and use (i) to show $\widehat{f_\gamma}$ is a function. Apply Problem 2 together with the scaling property $f_\gamma(tx) = t^{-\gamma} f_\gamma(x)$.
The case $\gamma < d/2$ follows by inverse transform, and case $\gamma = d/2$ by a limiting argument.]