**Department of Mathematics and Statistics**
**Fourier analysis**
**Exercise 10**
**Nov. 30, 2015**

1. (i) If $f_0(x) = |x|$, $x \in \mathbb{R}$, show that $f_0 \in \mathcal{S}'(\mathbb{R})$.
   (ii) Calculate the (distributional) derivatives $f_0', f_0''$. What is $x^2 f_0^{(4)}$ ?

2. Given $a \in \mathbb{R}^d$, let the distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ be given by

$$T(g) \;=\; \int_{-1}^{1} g(ta)\, dt.$$

(c.f. Exercises 9). Show that the distributional derivative $\sum_{j=1}^d a_j \partial_j T$ can be expressed a sum of suitable $\delta$-functions.

3. (i) Does $|a|\delta_a \to 0$ converge in $\mathcal{S}'(\mathbb{R})$, when $a \to \infty$ ? If yes, what is the limit ?
   (ii) If $f_\varepsilon(x) = e^{-\varepsilon|x|}$, $x \in \mathbb{R}$, show that $\widehat{f_\varepsilon}(\xi) = \dfrac{2\varepsilon}{\varepsilon^2 + \xi^2}$. Find the distribution $T \in \mathcal{S}'(\mathbb{R})$ for which $f_\varepsilon \to T$ in $\mathcal{S}'$ as $\varepsilon \to 0$. What can you say of the Fourier transform of the limit distribution $T$ ? Can you show directly that $\widehat{f_\varepsilon} \to \widehat{T}$ in $\mathcal{S}'$ ?

4. Suppose that $\phi \in C^\infty(\mathbb{R}^d)$ and that for every $\alpha \in \mathbb{N}$ we can find constants $C, M < \infty$ such that $|\partial^\alpha \phi(x)| \leq C(1 + |x|^2)^M$, $x \in \mathbb{R}^d$. Show that

$$g \mapsto \phi g$$

is a continuous map from $\mathcal{S}(\mathbb{R}^d)$ onto itself, and that $\phi T \in \mathcal{S}'(\mathbb{R}^d)$ whenever $T \in \mathcal{S}'(\mathbb{R}^d)$.

5. (i) Let $g \in \mathcal{S}(\mathbb{R})$ and write $g_\varepsilon(x) := \varepsilon^{-1}\big(g(x+\varepsilon) - g(x)\big)$. Show that $g_\varepsilon(x) \to g'(x)$ in the metric of $\mathcal{S}(\mathbb{R})$ as $\varepsilon \to 0$.
   (ii) Using the above, show that for every $f \in L^1(\mathbb{R})$ we have, as $\varepsilon \to 0$,

$$\varepsilon^{-1}\big(f(x+\varepsilon) - f(x)\big) \to \frac{d}{dx} f \qquad \text{in } \mathcal{S}'(\mathbb{R}),$$

where $\frac{d}{dx} f$ is the distributional derivative of $f$.