Department of Mathematics and Statistics
Fourier analysis
Exercise 1
Sept. 7, 2015

1. Show with help of Euler's formula, that

$$\int_0^L \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{L}{2}\delta_{k,n}$$

2. a) Let $f(x) = \sum_{n=-N}^{N} c_n e^{inx}$ be a trigonometric polynomial. Use Euler's formula to show that equivalently, f can be written in the form

$$f(x) = a_0 \;+\; \sum_{n=1}^{N} a_n \cos(nx) \;+\; \sum_{n=1}^{N} b_n \sin(nx). \tag{1}$$

Determine a_n, b_n in terms of the coefficients c_n. And conversely.

b) Show that the above coefficients a_n and b_n can be obtained from the formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)f(x)dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)f(x)dx, \qquad n = 1,\ldots,N,$$

$$\text{with} \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

3. a) If f is odd, that is $f(x) \equiv -f(-x)$, show that $\widehat{f}(n) = -\widehat{f}(-n)$ and that the Fourier series (1) consists of sin-functions only.

b) If $g : [x_0, x_0 + L] \to \mathbb{C}$ is given, define $f(x) = g\left(\frac{L}{2\pi}x + x_0\right)$, $x \in [0, 2\pi]$. Show that

$$\widehat{g}(n) = \widehat{f}(n)\, e^{-i\frac{2\pi n}{L}x_0}, \qquad n \in \mathbb{Z}.$$

4. Suppose $f(x)$ is the 2π-periodic function defined by $f(x) = \pi - |x|$, for $x \in [-\pi, \pi]$. Show that the Fourier coefficients of f are given by

$$\widehat{f}(n) = \begin{cases} \pi/2 & \text{if } n = 0 \\ \frac{2}{\pi n^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even, } n \neq 0. \end{cases}$$

5. a) Suppose that $f(x)$ is the function of Problem 4. Show using results from the lectures, that at every point $x \in [-\pi, \pi]$ the Fourier series of f converges to $f(x)$.

b) Evaluate the series at $x = 0$ and show that

$$1 + 3^{-2} + 5^{-2} + \ldots \;=\; \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \;=\; \frac{\pi^2}{8}.$$