Exercise 1 (chapter 3.4): Let the random variable X follow a Normal distribution with mean μ and variance σ^2. The density $g_X(x)$ of the truncated Normal distribution with support on the interval $I = (a, b)$ and $a < b$ is

$$g_X(x) = \frac{\phi(x; \mu, \sigma^2)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} 1_{(a,b)}(x) .$$

Derive the inverse transformation method to simulate from the truncated Normal distribution.

Exercise 2 (chapter 3.5): Let $p^*(\theta \,|\, y)$ be an unnormalized posterior density of a parameter Θ and $g(\theta)$ the density of the proposal distribution in the accept-reject method. The conditional acceptance probability of a proposed value θ' is

$$\Pr(\text{accepted} \,|\, \theta') = \frac{p^*(\theta' \,|\, y)}{M g(\theta')} ,$$

where $M > 0$ is a known majorizing constant such that $p^*(\theta \,|\, y) \leq M g(\theta)$ for all θ. Derive the unconditional acceptance probability. For univariate distributions, this happens to be equal to

$$\frac{\text{Area under } p^*(\theta \,|\, y)}{\text{Area under } M g(\theta)} .$$

Exercise 3 (chapter 3.5): Let $f_X(x)$ be the density of a random variable X with support on the interval $I = [a, b]$. Consider the special version of the accept-reject method:

1. Generate independently u_1 and u_2 from standard Uniform distributions.
2. Accept $x' = a + (b - a)u_1$ as a sample from the distribution of X if

   $$M u_2 \leq f_X(a + [b - a]u_1) ,$$

   where $M = \max_{a \leq x \leq b} f_X(x)$.

Derive the unconditional acceptance probability and demonstrate why x' is a sample from the distribution of X with cumulative distribution function

$$F_X(x) = \int_a^x f_X(t) \, \mathrm{d}t .$$

Exercise 4 (chapter 3.5): Let the random variable X and Y be independent and follow Uniform distributions on the interval $I = (-0.5, 0.5)$. The density of the random variable $W = X + Y$ is known as the triangular density

$$f_W(w) = 1 - |w| , \quad |w| < 1 .$$

with support on the interval $I = (-1, 1)$. Use the special accept-reject method to simulate from the distribution of W.

Exercise 5 (chapter 3.5):

1. Let the random variable X follow a standard Normal distribution. The density of the standard Normal distribution is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}.$$

Determine the value of the majorizing constant M in the accept-reject method using a standard Cauchy distribution as the proposal distribution. The density of the standard Cauchy distribution is

$$f_X(x) = \frac{1}{\pi(1+x^2)}.$$

2. Demonstrate that it is not possible to simulate from the standard Cauchy distribution using the accept-reject method with a standard Normal proposal distribution.

Exercise 6 (chapter 3.5):

1. Suppose that it is possible to compute the maximum likelihood estimate $\hat{\theta}_{\mathrm{MLE}}$ of a parameter Θ, that is,

$$\hat{\theta}_{\mathrm{MLE}} \subseteq \left\{ \operatorname*{argmax}_{\theta \in \Theta} p(y \mid \theta) \right\}.$$

Show that the prior of Θ can be used as the proposal distribution in the accept-reject method to simulate from the posterior with unnormalized density

$$p^*(\theta \mid y) = p(y \mid \theta) p(\theta).$$

Derive the acceptance condition of the accept-reject method.

2. Suppose that the likelihood $p(y \mid \theta)$ can be normalized to yield a so-called normalized likelihood. Let the prior density $p(\theta)$ be bounded and assume that direct simulation from the normalized likelihood is feasible. Show that the normalized likelihood of Θ can be used as the proposal distribution in the accept-reject method to simulate from the posterior. Derive again the acceptance condition of the accept-reject method.