This assignment contains 4 problems. You *may* use your books and notes. You are required to show your work on each problem of this assignment. The following rules apply:

- **No corrections will be made once the assignment is distributed.** If there is a typo or ambiguity in a question, state your assumption and answer accordingly.
- **Organize your report** ("fullname_report_COMPSTAT2015.pdf") in a reasonably neat and coherent way. Algebraic work and source code scattered all over without a clear ordering will receive very little credit. Provide a title page with your report that contains your full name, email address and student number. Put the same information on top of every page of your report, in case the pages become separated.
- **All R source code files need to be handed in** and named "fullname_code1_COMPSTAT2015.R", "fullname_code2_COMPSTAT2015.R", etc. Put your full name, email address and student number in the source code header comments.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, source code or algebraic work will receive no credit. An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you are graduating or returning to your home university at the end of the semester, please write "GRADUATING" or "EXCHANGE" on the title page of your report.
- **The deadline for returning your results is Sunday, January 31, 2016**. ZIP compress all files and folders and email the archive named "fullname_results_COMPSTAT2015.zip" to christian.user@helsinki.fi.

Task: Let $\{Y_i\}_{i=1}^n$ be independent and identically distributed Gamma random variables with shape $\alpha > 0$ and rate parameter $\beta > 0$. The density of the Gamma distribution is

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp\{-\beta y\} \qquad x > 0\,.$$

The priors on α and β are Exponential distributions with rate parameter $\lambda = 0.001$. The density of Exponential distribution is

$$f_X(x) = \lambda e^{-\lambda x} \qquad x > 0\,.$$

Generate $n = 50$ realizations of independent and identically distributed Gamma random variables $\{Y_i\}_{i=1}^n$ by using the following R code:

```
alpha <- 7 ; beta <- 9 ; nY <- 50
set.seed( 100 )
y <- rgamma( nY, alpha, beta )
```

The assignment is to perform inference about the parameters α and β by means of different MCMC samplers.

Problem 1: Derive the maximum likelihood estimator of α and β. Identify the full conditionals $p(\alpha \,|\, \beta, y)$ and $p(\beta \,|\, \alpha, y)$. Do they correspond to familiar distributions?

Problem 2: Implement a random walk Metropolis–Hastings sampler for the constrained parameters $\alpha, \beta > 0$. Immediately reject proposed values if they are not both positive. Select the covariance matrix $a\boldsymbol{\Sigma}$ of the proposal distribution so that the acceptance rate becomes reasonable (10–40%). Report posterior summary statistics and numerical standard errors, produce trace, autocorrelation and cumulative average plots for the parameters (α, β) and explain how you calibrated the tuning constant a.

Problem 3: Consider the following change–of–variables

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = g(\alpha, \beta) = \begin{bmatrix} \log \alpha \\ \log(\alpha/\beta) \end{bmatrix},$$

where $\mu = \alpha/\beta$ is the expected value of a random variable that follows a Gamma(α, β) distribution. Show that (α, μ) and therefore (ϕ, ψ) are orthogonal parameters by demonstrating that the expected Fisher information matrix $\mathcal{I}(\alpha, \mu)$ is diagonal.[1]

Implement a random walk Metropolis–Hastings sampler for the unconstrained parameters (ϕ, ψ). Select the covariance matrix $a\boldsymbol{\Sigma}$ of the proposal distribution so that the acceptance rate becomes reasonable (10–40%). Report posterior summary statistics and numerical standard errors, produce trace, autocorrelation and cumulative average plots for the original parameters (α, β) and explain how you calibrated the tuning constant a.

Problem 4: Implement a hybrid sampler by taking advantage of conditional conjugacy. Update $\phi = g(\alpha) = \log \alpha$ with a random walk Metropolis–Hasting step and β by drawing from its full conditional distribution conditionally on the proposed value of α. Accept or reject the proposed pair jointly by using the ordinary Metropolis–Hastings acceptance rule. Tune the proposal distribution so that the acceptance rate becomes reasonable (10–40%). Report posterior summary statistics and numerical standard errors, produce trace, autocorrelation and cumulative average plots for the parameters (α, β) and explain how you calibrated the tuning constant.

[1]Given a large sample size, ϕ and ψ are then approximately independent in the posterior regardless of the prior because the posterior is approximately normal with covariance matrix given by the inverse of the expected Fisher information matrix.