FINITE DIFFERENCE SOLUTION OF POISSON'S EQUATION

Samuli Siltanen, December 2, 2015

This is reading material for the course *Applications of matrix computations* given in the fall of 2015 at University of Helsinki.

Contents

1. Introduction

We discuss the solution of the Poisson equation

$$\Delta u = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad \text{in } \Omega \subset \mathbb{R}^2 \tag{1}$$

with Dirichlet boundary condition

$$u_{\partial\Omega} = f, \tag{2}$$

where f is a given function.

The physical interpretation of (1) and (2) is the following. We maintain the temperature distribution f on the boundary $\partial\Omega$ of a heat-conducting two-dimensional body Ω, such as a metal plate of constant thickness. Over a long period of time, the metal plate settles down in a steady-state temperature distribution that does not change significantly in time anymore. That steady-state temperature distribution is given by $u(x,y)$.

The Dirichlet problem (1) and (2) is an example of an elliptic partial differential equation (PDE). For more information about the theory of PDEs, see [1, 2, 3, 5, 6].

Our computational solution method of choice here is the Finite Fifference (FD) method. Instead of doing general theory or algorithm development here, we focus on a simple case study. For a great treatment of numerical solution of PDEs, see [4].

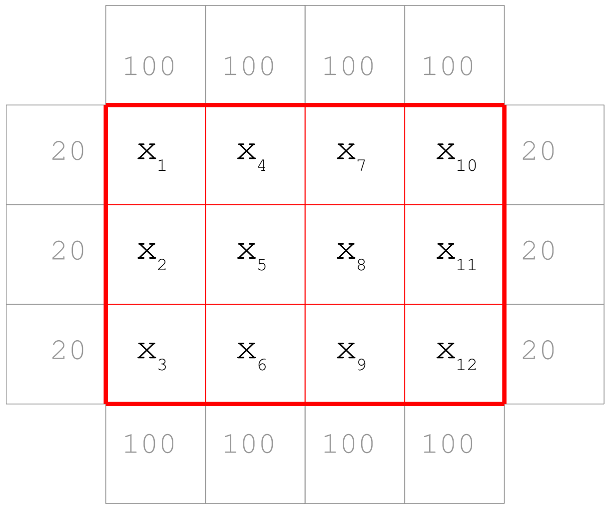

FIGURE 1. Rectangular image area, shown with a thick red line, discretized by dividing into 3×4 = 12 pixels. Note the one-index numbering of the pixels following the Matlab convention.

2. DISCRETE MODELS FOR u AND f

The function $u = u(x, y)$ is defined inside the rectangle Ω. We divide the rectangle into 3×4 pixels and number the pixels x_k using just one integer index $k = 1, \ldots, 12$. See Figure 1. Thus our unknown can be written as a vector:

$$x = [x_1, x_2, \ldots, x_{12}]^T. \tag{3}$$

Denote the number of rows by R (here $R = 3$) and the number of columns by C (here $C = 4$). The index of the pixel located in row i and column j can be computed with the formula

$$k = (j - 1)R + i. \tag{4}$$

The boundary data f is represented discretely as values in "extra pixels" outside the actual image domain. In this simple example we think that the top and bottom boundaries are kept at a temperature of 100°C and the left and right boundaries at 20°C.

3. FD approximation to the Laplace operator

Assume first that a function $g(x)$ of one real variable is sampled on uniformly distributed grid points with spacing Δx between them. Then a finite difference approximation for the derivative of g is given by

$$g'(x) \approx \frac{g(x+\Delta x)-g(x)}{\Delta x}.$$

Similarly, we can approximate the second derivative of g as

$$g''(x) \approx \frac{g(x-\Delta x)-2g(x)+g(x+\Delta x)}{(\Delta x)^2}.$$

Now consider the function $u(x,y)$ of two variables. We can use the above finite difference approximations for partial derivatives in both x and y. This leads to the expression

$$\begin{aligned}\Delta u &= \frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}\\ &\approx \frac{1}{(\Delta x)^2}\Big(-4u(x,y)+u(x-\Delta x,y)+u(x+\Delta x,y)+\\ &\qquad +u(x,y-\Delta y)+u(x,y+\Delta y)\Big).\end{aligned} \tag{5}$$

Now formula (5) means that $\Delta u(x,y)=0$ is approximately equivalent to demanding that the temperature $u(x,y)$ is the average of the temperatures at four neighboring points (up, down, left and right by steps of length Δx).

The equation (5) relates to our discrete model as follows. Consider the five-point stencil shown in Figures 2, 3 and 4. In the situation of Figure 2, the equation (1) and boundary condition (2) are approximated by requiring that the pixel value x_1 is the average of the pixel values x_2 and x_4 as well as the boundary data 100 and 20. As an equation this can be expressed as $x_1=(x_2+x_4+100+20)/4$.

In the situation of Figure 3, the equation (1) and boundary condition (2) are approximated by requiring that the pixel value x_2 is the average of the pixel values x_1, x_3 and x_5 as well as the boundary data 20. As an equation this can be expressed as $x_2=(x_1+x_3+x_5+20)/4$.

In the situation of Figure 4, the equation (1) and boundary condition (2) are approximated by requiring that the pixel value x_5 is the average of the pixel values x_2, x_4, x_6 and x_8. As an equation this can be expressed as $x_2=(x_1+x_3+x_5+20)/4$.

Since this is a course on applications of matrix computations, our goal is to write equation (1) and boundary condition (2) in the form of a matrix equation

$$Ax=m, \tag{6}$$

where $x\in\mathbb{R}^{12}$ is as in (3). The 12×12 matrix A and the right hand side vector $m\in\mathbb{R}^{12}$ must be so designed that they implement twelwe linear equations for the variables x_j arising from (1) and (2) as shown above. We get $m = [120, 20, 120, 100, 0, 100, 100, 0, 100, 120, 20, 120]^T$ and

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}.$$

In this low-resolution case it is easy to solve the equation (6) as $x = A^{-1}m$. For the result, see Figure 5. The actual pixel values in the solution are

62.8169	77.4648	77.4648	62.8169
53.8028	69.5775	69.5775	53.8028
62.8169	77.4648	77.4648	62.8169

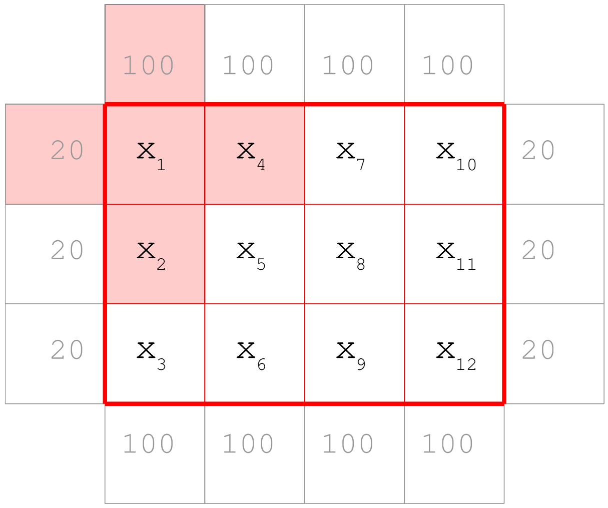

FIGURE 2. Five-point stencil centered at the corner.

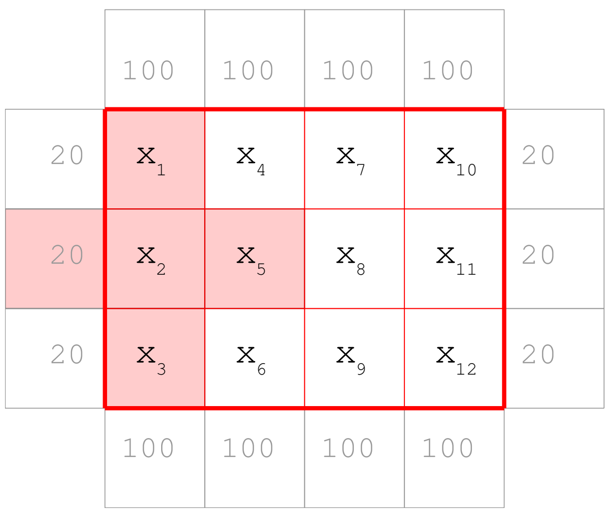

FIGURE 3. Five-point stencil centered at a side pixel.

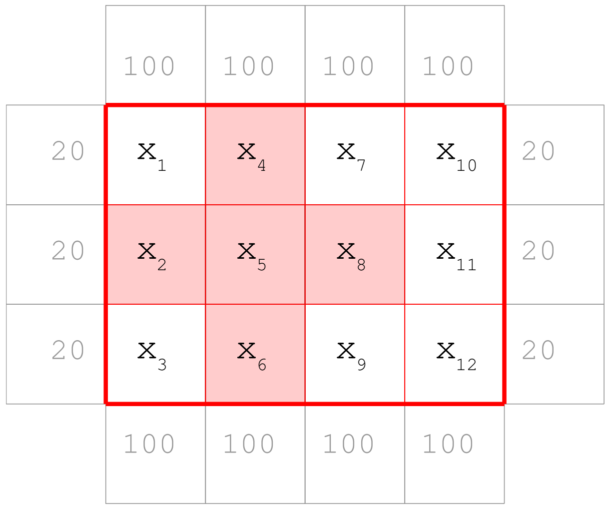

FIGURE 4. Five-point stencil centered at a central pixel.

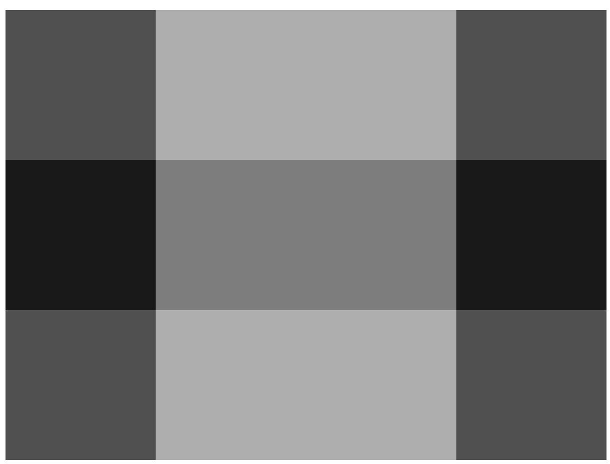

FIGURE 5. Temperature distribution $u(x, y)$ solved in a $3{\times}4$ grid.

4. A higher-resolution computation

To see a more detailed solution we need many rows and columns in the discretized image. This means that the unknown $x \in \mathbb{R}^n$ with a large n. In this case thechances are that attempting to solve the linear system (6) by $x = A^{-1}m$ will use up all the memory in the computer. This is because computing the inverse of a large matrix is computationally demanding.

However, we can do some tricks. First of all, the matrix A can be constructed and stored as a *sparse matrix*. Since most of the elements of A are zero, the sparse representation uses very little memory as it only records the row and column indices (and values) of nonzero elements of A. Furthermore, we do not need to compute the inverse A^{-1} for solving the equation $Ax = m$. Approximate solution can be computed using so-called *iterative Krylov subspace methods*, such as GMRES. These methods only need an efficient algorithm for computing the vector Av for any given vector $v \in \mathbb{R}^n$. For sparse matrices it is very easy to compute Av.

See Figure 6 for the solution of (1) and (2) computed using a grid with 150 rows and 200 columns.

References

[1] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[2] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[3] F. John. *Partial differential equations*. Springer-Verlag, New York-Berlin, 1971. Applied Mathematical Sciences, Vol. 1.

[4] Stig Larsson and Vidar Thomée. *Partial differential equations with numerical methods*, volume 45 of *Texts in Applied Mathematics*. Springer-Verlag, Berlin, 2003.

[5] Michael Renardy and Robert C Rogers. *An introduction to partial differential equations*, volume 13. Springer Science & Business Media, 2006.

[6] Michael E. Taylor. *Partial differential equations I. Basic theory*, volume 115 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.

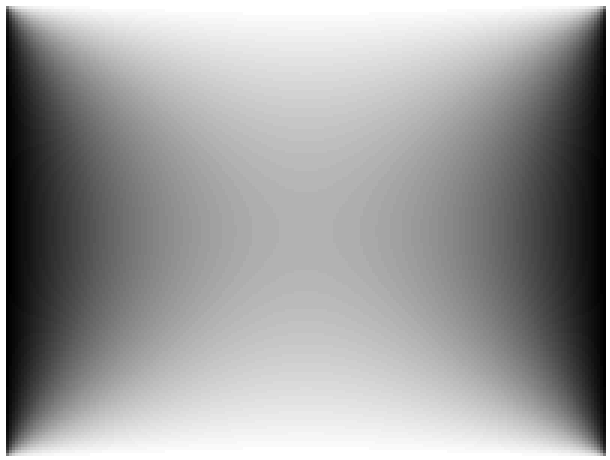

FIGURE 6. Temperature distribution $u(x, y)$ solved in a 150×200 grid.