BRIEF INTRODUCTION TO FOURIER SERIES

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Parametrize the boundary of the unit circle as

$$\{(\cos\theta, \sin\theta) \,|\, 0 \leq \theta < 2\pi\}.$$

We will use the *Fourier basis functions*

$$\varphi_n(\theta) = (2\pi)^{-1/2} e^{in\theta}, \quad n \in \mathbb{Z}. \tag{1}$$

We can approximate 2π-periodic functions $f : \mathbb{R} \to \mathbb{R}$ following the lead of the great applied mathematician Joseph Fourier (1768–1830). Define cosine series coefficients using the L^2 inner product

$$\widehat{f}_n := \langle f, \varphi_n \rangle = \int_0^{2\pi} f(\theta)\, \overline{\varphi_n(\theta)}\, d\theta, \qquad n \in \mathbb{Z}.$$

Then, for nice enough functions f, we have

$$f(\theta) \approx \sum_{n=-N}^{N} \widehat{f}_n \, \varphi_n(\theta)$$

with the approximation getting better when N grows.

Note that the functions φ_n are orthogonal:

$$\langle \varphi_n, \varphi_n \rangle = \delta_{nm}.$$