Stochastic analysis, fall 2014, Final Exam

1. Let $(B_t^{(1)}, \dots, B_t^{(n)} : t \ge 0)$ continuous local martingales in the filtration $\mathbb{F}$ with

$$\langle B^{(i)}, B^{(i)} \rangle_t = t,$$
$$\langle B^{(i)}, B^{(j)} \rangle_t = E_P\big(B_t^{(i)} B_t^{(j)}\big) = c_{ij} t, \text{ for } i \neq j, .$$

with $c_{ij} \in [-1, 1]$ constant.

   (a) Each $B_t^{(i)}$ is a Brownian motion. Why ?

   (b) Assume $B_0^{(i)} = 0$ at time $t = 0$.
   Use inducively Ito formula and Fubini Theorem to compute the joint moment at time t:

$$E_P\big(B_t^{(1)} \dots B_t^{(n)}\big) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ t^{n/2} \sum\limits_{\text{pairings}} \prod\limits_{pairs\{i,j\}} c_{ij} & \text{if } n \text{ is even} \end{cases}$$

   where when n is even, the sum is over all pairings of $1, \dots, n$ into $n/2$ pairs, where the pairs are disjoint and the elements of the pairs are distince. For each pairing we then take the product over the pairs of the pairing.
   Hint: Compute the semimartingale decomposition of the product $B_t^{(1)} \dots B_t^{(n)}$, and show that the local martingale is a true martingale (which therefore has zero expectation).
   This is Wick's formula (in the literature usually the proof is based on the moment generating function).

2. Let (B_t) be a standard Brownian motion, denote $i = \sqrt{-1}$ as usual. Recall that

$$Z(t, \theta) = \exp\left(i\theta B_t + \frac{1}{2}\theta^2 t\right) =$$
$$\cos(\theta B_t) \exp(\theta^2 t/2) + i \sin(\theta B_t) \exp(\theta^2 t/2) = M_t(\theta) + i N_t(\theta)$$

is a complex valued martingale $\forall \theta \in \mathbb{R}$, that is both real and imaginary parts are martingales.
Compute the brackets $\langle M(\theta), M(\theta) \rangle_t$, $\langle N(\varphi), N(\varphi) \rangle_t$, $\langle M(\theta), N(\varphi) \rangle_t$.

3. In the setting of exercise 2,
Compute the Ito-Clarck martingale representation of the square integrable random variable

$$X_T = \sin(\theta B_T) \cos(\varphi B_T) = E\big(\sin(\theta B_T) \cos(\varphi B_T)\big) + \int_0^T Y_s dB_s$$

i.e. compute the expectation and find the adapted integrand process Y_s.

Hint. rewrite

$$X_T = cM_T(\theta)N_T(\varphi)$$

with $c = \exp(-(\theta^2 + \varphi^2)T/2)$, and use integration by parts, to find the martingale decomposition of the product $(M_t(\theta)N_t(\varphi))$.

4. Let $X_T = \exp(\theta B_T)B_T^2$, where $\theta \in \mathbb{R}$.

   a) Show that $X_T \in L^2(\Omega)$.

   b) Compute $E(X_T)$.

   c) Compute the Ito-Clarck martingale representation of X_T. Hint: use Ito formula and integration by parts.

5. (a) Solve the following Ito SDE

$$a) \quad X_t = x + \int_0^t \sqrt{1 - X_s^2}dB_s - \frac{1}{2}\int_0^t X_s ds$$

$$b) \quad X_t = x + \int_0^t \sqrt{1 + X_s^2}dB_s + \frac{1}{2}\int_0^t X_s ds$$

$$c) \quad X_t = x + \int_0^t \sqrt{1 + X_s^2}dB_s + \int_0^t \big(\sqrt{1 + X_s^2} + \frac{1}{2}X_s\big)ds$$

$$b) \quad X_t = x + \int_0^t \exp(-X_s)dB_s + \frac{1}{2}\int_0^t \exp(-2X_s)ds$$

$$c) \quad X_t = x + \frac{1}{3}\int_0^t (X_s)^{1/3}ds + \int_0^t (X_s)^{2/3}dB_s$$

   Hint: assume that $X_t = \varphi(B_t)$ and use Ito formula to obtain an equation for φ.
   In c) you can assume first that $X_t = \varphi(B_t + a(t))$ and after using Ito formula, choose the function $a(t)$ to simplify the differential equation for φ.

   (b) Rewrite the SDE in Stratonovich form.

   Remark in general is not always possible to find an explicit solutions of a SDE.

6. Let $B^{(1)}$ and $B^{(2)}$ two independent Brownian motions under the measure P and let

$$X_t = x^{(0)}t + x^{(1)}B_t^{(1)} + x^{(2)}B_t^{(2)}$$
$$Y_t = y^{(0)}t + y^{(1)}B_t^{(1)} + y^{(2)}B_t^{(2)}$$

   where $x^{(i)}, y^{(i)}$ are deterministic constants, $i = 0, 1, 2$.

   Using Girsanov theorem, construct a probability measure Q equivalent to P on finite intervals $[0, t]$ such that both X_t and Y_t are Q-martingales.

   Under which conditions on the coefficients $x^{(i)}, y^{(i)}$ such Q is unique ?

7. We consider a family of linear SDE in Ito sense

$$X_t = x + \int_0^t X_s\theta ds + \int_0^t X_s\sigma dB_s^\theta$$

where (B_t^θ) is Brownian motion under the measure P^θ. We think as $\sigma \neq 0$ fixed, while $\theta \in \mathbb{R}$ is a parameter. Note that

$$B_t^\theta = B_t^0 - \frac{\theta}{\sigma}\, t$$

where B_t^0 is a Brownian motion under P^0 which corresponds to the value $\theta = 0$.

a) Compute and the likelihood ratio process

$$Z_t(\theta) = \frac{dP_t^\theta}{dP_t^0}$$

and find a representation as stochastic integral with respect to the integrator (X_t).

b) Show that $Z_t(\theta)$ is a martingale under P^0.

c) Compute the logarithmic derivative

$$S_t(\theta) := \frac{d}{d\theta} \log Z_t(\theta)$$

and show that $S_t(\theta)$ is a martingale under P^θ.

d) Assuming now that the parameter θ is unknown, compute the maximum likelihood estimator $\hat{\theta}_T$ for a given a realization $(X_t(\omega) : t \in [0, T])$. In other words, find the argument $\hat{\theta}_T$ which maximizes $\log(Z_T(\theta, \omega))$ for the observed realization.