Evolution and the Theory of Games
Fall 2014
Exercises 5-8

5 Find all Nash equilibria (mixed and pure) of the Hawk-Dove game for R>C and for R<C.

	H	D
H	$\frac{R-C}{2}, \frac{R-C}{2}$	$R, 0$
D	$0, R$	$\frac{R}{2}, \frac{R}{2}$

Pure strategy Nash equilibria:

$\boxed{R>C}$

$\pi_1(D,H) = 0 < \frac{R-C}{2} = \boxed{\pi_1(H,H)}$
$\pi_1(D,D) = \frac{R}{2} < R = \pi_1(H,D)$

$\pi_2(H,D) = 0 < \frac{R-C}{2} = \boxed{\pi_2(H,H)}$
$\pi_2(D,D) = \frac{R}{2} < R = \pi_2(D,H)$

$\Rightarrow$ (H,H) is the only Nash equilibrium when R>C.

$\boxed{R<C}$

$\pi_1(H,H) = \frac{R-C}{2} < 0 = \boxed{\pi_1(D,H)}$
$\pi_1(D,D) = \frac{R}{2} < R = \boxed{\pi_1(H,D)}$

$\pi_2(H,H) = \frac{R-C}{2} < 0 = \boxed{\pi_2(H,D)}$
$\pi_2(D,D) = \frac{R}{2} < R = \boxed{\pi_2(D,H)}$

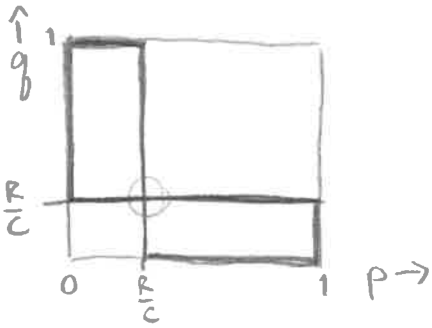
$\Rightarrow$ (D,H) & (H,D) are Nash equilibria when R<C.

Mixed strategy Nash equilibria:

Let $x = (p, 1-p)$ & $y = (q, 1-q)$

$$\pi_1(x,y) = pq\left(\frac{R-C}{2}\right) + p(1-q)R + (1-p)q\cdot 0 + (1-p)(1-q)\frac{R}{2}$$
$$= pq\frac{R}{2} - pq\frac{C}{2} + pR - pqR + \frac{R}{2} - p\frac{R}{2} - q\frac{R}{2} + pq\frac{R}{2}$$
$$= pq\left(R - \frac{C}{2} - R\right) + p\frac{R}{2} - q\frac{R}{2} + \frac{R}{2}$$
$$= (1-q)\frac{R}{2} + p\frac{R}{2} - pq\frac{C}{2}$$
$$= (1-q)\frac{R}{2} + \frac{1}{2}(R - qC)\,p$$

If $R - qC > 0$ (i.e. $q < \frac{R}{C}$), x should play $p = 1$.
If $R - qC < 0$ (i.e. $q > \frac{R}{C}$), x should play $p = 0$
If $R - qC = 0$ (i.e. $q = \frac{R}{C}$), x's strategy makes no difference.



$$\pi_2(x,y) = pq\left(\frac{R-C}{2}\right) + p(1-q)\cdot 0 + (1-p)qR + (1-p)(1-q)\frac{R}{2}$$

$$= pq\cdot\frac{R}{2} - pq\cdot\frac{C}{2} + qR - pqR + \frac{R}{2} - p\frac{R}{2} - q\frac{R}{2} + pq\frac{R}{2}$$

$$= \cancel{pqR} - \cancel{pqR} + q\frac{R}{2} - pq\frac{C}{2} + \frac{R}{2} - p\frac{R}{2}$$

$$= (1-p)\frac{R}{2} + \left(\frac{R}{2} - p\frac{C}{2}\right)q$$

$$= (1-p)\frac{R}{2} + \frac{1}{2}(R - pC)q$$

If $p > \frac{R}{C}$, Y should play $q = 0$.
If $p < \frac{R}{C}$, Y should play $q = 1$.
If $p = \frac{R}{C}$, Y can choose any strategy.

So, we find the Nash equilibrium $\begin{cases} \hat{x} = \left(\frac{R}{C}, 1 - \frac{R}{C}\right) \\ \hat{y} = \left(\frac{R}{C}, 1 - \frac{R}{C}\right) \end{cases}$

Note that this equilibrium exists only when $\frac{R}{C} < 1$, i.e. when $\boxed{R < C}$.

Note: The top-left and bottom-right corners of the swastika diagram show the Nash equilibria $\begin{cases} \hat{x} = (0,1) \\ \hat{y} = (1,0) \end{cases}$ and $\begin{cases} \hat{x} = (1,0) \\ \hat{y} = (0,1) \end{cases}$, which correspond to the pure-strategy Nash equilibria, (D,H) and (H,D), respectively.

$\boxed{6}$ Suppose that $(\hat{x},\hat{y})$ is a Nash equilibrium. Show that $\pi_1(x,\hat{y}) = \pi_1(\hat{x},\hat{y})$ for every pure strategy x in the support of $\hat{x}$.

- Let $\hat{x} = (p_1,\dots,p_n)$, where p_i is the probability of playing pure strategy x_i.

- As $(\hat{x},\hat{y})$ is a Nash equilibrium,

$$\pi_1(x_i,\hat{y}) \le \pi_1(\hat{x},\hat{y}) \quad \text{for } x_i \in \mathbb{X},\ i = 1,\dots,n$$

(Assume $\mathbb{X}$ has n elements.)

- Assume the inequality is strict, i.e. assume $\pi_1(x_i,\hat{y}) < \pi_1(\hat{x},\hat{y})$ for some x_i in the support of $\hat{x}$, and assume that $\pi_1(x_j,\hat{y}) = \pi_1(\hat{x},\hat{y})$ $\forall j \ne i$ (where x_j is in the support of $\hat{x}$).
Then, $\pi_1(\hat{x},\hat{y}) = \sum_j p_j \pi_1(x_j,\hat{y}) = p_i\pi_1(x_i,\hat{y}) + \sum_{j\ne i} p_j \pi_1(x_j,\hat{y})$

$$< p_i\pi_1(\hat{x},\hat{y}) + \sum_{j\ne i} p_j\pi_1(x_j,\hat{y}) = \pi_1(\hat{x},\hat{y}),$$

which is impossible. Thus, for no x_i in the support of $\hat{x}$ is it possible that $\pi_1(x_i,\hat{y}) < \pi_1(\hat{x},\hat{y})$. So, $\pi_1(x,\hat{y}) = \pi_1(\hat{x},\hat{y})$ $\forall x$ in the support of $\hat{x}$, QED.

$\boxed{7}$ Show that every dominating strategy sol'n is a Nash equilibrium, but that the reverse is not necessarily true.

- $x \in \mathbb{X}$ is dominated by $x' \in \mathbb{X}$ if $\pi_1(x,y) \le \pi_1(x',y)$ $\forall y \in \mathbb{Y}$ with a strict inequality for at least one $y \in \mathbb{Y}$.
- $(\hat{x},\hat{y}) \in \mathbb{X}\times\mathbb{Y}$ is NE provided that

$$\begin{cases} \pi_1(x,\hat{y}) \le \pi_1(\hat{x},\hat{y}) & \forall x \in \mathbb{X} \\ \pi_2(\hat{x},y) \le \pi_2(\hat{x},\hat{y}) & \forall y \in \mathbb{Y} \end{cases}$$

Assume that $\exists$ DSS (x_j, y_k) which is <u>not</u> NE. This means that $\exists\, x_i \ni \pi_1(x_i,y_k) > \pi_1(x_j,y_k)$ [or that $\exists\, y_i \ni \pi_2(x_j,y_i) > \pi_2(x_j,y_k)$]. Without loss of generality, assume the former. In this case, row i of the payoff matrix can never be dominated by row j in the iterated removal process, meaning that this process cannot end with only the strategy pair (x_j,y_k) left. As this contradicts the assumption that (x_j,y_k) is a DSS, it is impossible that (x_j,y_k) be a DSS but not a NE.

For an example of a NE that is not a DSS, consider the H-D game with $R<C$. From exercise 5, we know that (H,D) + (D,H) are NE, but neither is a DSS.

	H	D
H	$\frac{1}{2}(R-C), \frac{1}{2}(R-C)$	$R, 0$
D	$0, R$	$\frac{R}{2}, \frac{R}{2}$

$\boxed{8}$ Show that if $x \in \mathbb{X}$ is a <u>strictly</u> dominated pure strategy and $(\hat{x}, \hat{y}) \in \mathbb{X} \times \mathbb{Y}$ is NE, then x cannot be in the support of $\hat{x}$.

Show that this conclusion need not be true if x is only <u>weakly</u> dominated, using the following example payoff matrix.

	y_1	y_2	y_3
x_1	3,2	3,0	2,2
x_2	1,0	3,3	0,3
x_3	0,2	0,0	3,2

Suppose that x is in the support of $\hat{x}$. If x is strictly dominated by $\bar{x}$ then $\pi_1(x,y) < \pi_1(\bar{x},y)$ $\forall y \in \mathbb{Y}$. It follows immediately that $\pi_1(x,\hat{y}) < \pi_1(\bar{x},\hat{y})$. Since x is in the support of $\hat{x}$, by the Bishop-Cannings theorem $\pi_1(x,\hat{y}) = \pi_1(\hat{x},\hat{y}) < \pi_1(\bar{x},y)$, which contradicts that $(\hat{x},\hat{y})$ is NE. Thus, a strictly dominated pure strategy x cannot be in the support of $\hat{x}$.

In the above matrix, there are 3 NE:
(x_1, y_1), (x_2, y_2) + (x_3, y_3).
Notice that y_2 is <u>weakly</u> dominated by y_3 despite being in the support of a NE.
(y_1 is also weakly dominated by y_3, and x_2 is weakly dominated by x_1.)