Integral equations
Solutions to the fourth problem set

1. Define

$$x_+^a = \begin{cases} x^a, x > 0 \\ 0, x \leqslant 0. \end{cases}$$

Determine those values $a \in \mathbb{R}$ for which x_+^a has a weak derivative in the sense that we defined in the lectures.

Solution. Where a function is classically differentiable, weak derivative exists and coincides with the classical derivative. Thus, x_+^a is weakly differentiable in $\mathbb{R}_-$ with weak derivative 0, and weakly differentiable in $\mathbb{R}_+$ with weak derivative ax^{a-1}. Thus, only the weak differentiability near zero needs to be considered, and the weak derivative in $\mathbb{R}$, if it exists, can only be ax_+^{a-1}.

Weak differentiability requires local integrability. For $a \neq -1$, we have

$$\int_\varepsilon^1 x^a \, \mathrm{d}x = \frac{x^{a+1}}{a+1}\Big]_\varepsilon^{x=1} = \frac{1}{a+1} - \frac{\varepsilon^{a+1}}{a+1},$$

and the limit $\varepsilon \longrightarrow 0+$ exists and is finite if $a > -1$, and the integral $\int_\varepsilon^1$ tends to infinity when $a < -1$. When $a = -1$, we have

$$\int_\varepsilon^1 x^a \, \mathrm{d}x = \log x\Big]_\varepsilon^{x=1} = -\log \varepsilon,$$

and this tends to infinity as as $\varepsilon \longrightarrow 0+$. Thus, the function x_+^a is locally integrable exactly when $a > -1$.

The weak derivative needs to be locally integrable as well, and, by the above considerations, the function ax_+^{a-1} is locally integrable if and only if $a = 0$ or $a - 1 > -1$. In other words, ax_+^{a-1} is locally integrable exactly when $a \geqslant 0$.

Thus, x_+^a can be weakly differentiable only when $a \geqslant 0$, so suppose then, that $a \geqslant 0$. The only remaining requirement for weak differentiability is that we need to have

$$\int_0^\infty x^a \, \varphi'(x) \, \mathrm{d}x = -\int_0^\infty ax^{a-1} \, \varphi(x) \, \mathrm{d}x$$

for all test functions $\varphi \in C_c^\infty(\mathbb{R})$. This requirement simplifies to

$$\int_0^\infty \big(x^a \, \varphi(x)\big)' \, \mathrm{d}x = 0.$$

As φ is compactly supported, this holds exactly when $\varepsilon^a \, \varphi(\varepsilon)$ tends to zero as $\varepsilon \longrightarrow 0+$. For $a = 0$ the limit is $\varphi(0)$ and this might not vanish. For $a > 0$, the limit is exists and vanishes, and we conclude that x_+^a is weakly differentiable exactly when $a > 0$, and the weak derivative is then ax_+^{a-1}.

For the next three exercises we assume that H is a **real** Hilbert space. Especially, the inner product $\langle \cdot, \cdot \rangle$ is an $\mathbb{R}$-bilinear map on $H \times H$.

2. Assume that $B\colon H \times H \longrightarrow \mathbb{R}$ is a real bilinear map for which there exist constants $M > 0$ and $m > 0$ such that

$$|B(u,v)| \leqslant M \|u\| \|v\|, \quad u, v \in H,$$

and

$$m \|u\|^2 \leqslant B(u,u), \quad u \in H.$$

Prove that there is a unique bounded linear operator $A\colon H \longrightarrow H$ such that

$$B(u,v) = \langle Au, v \rangle, \quad u, v \in H.$$

Solution. For any given $u \in H$, the mapping $B(u, \cdot)$ is a bounded linear functional of H, and so, by Riesz's representation theorem, there exists a unique $w \in H$ such that

$$B(u,v) = \langle w, v \rangle$$

for all $v \in H$. Since w is unique, we may define a mapping $A\colon H \longrightarrow H$ by setting $Au = w$, and this mapping satisfies $B(u,v) = \langle Au, v \rangle$ for all $u, v \in H$, and it is the unique mapping with this property.

Let $\alpha, \alpha' \in \mathbb{R}$ and $u, u' \in H$. Since

$$\begin{aligned} \langle A(\alpha u + \alpha' u'), v \rangle = B(\alpha u + \alpha' u', v) &= \alpha B(u,v) + \alpha' B(u', v) \\ &= \alpha \langle Au, v \rangle + \alpha' \langle Au', v \rangle = \langle \alpha Au + \alpha' Au', v \rangle \end{aligned}$$

for all $v \in H$, the mapping A is linear. Finally, by Riesz's representation theorem and the upper bound for $B(\cdot,\cdot)$, we have $\|Au\| = \|w\| \leqslant M \|u\|$, and so A is bounded.

3. Prove that the operator A constructed above is a bijection.

Solution. Given a vector $u \neq 0$ in H, we have

$$m \|u\|^2 \leqslant B(u,u) = \langle Au, u \rangle \leqslant \|Au\| \|u\|,$$

so that $\|Au\| \geqslant m \|u\| > 0$. Thus $Au \neq 0$ and we conclude that A is injective.

If $v \perp \operatorname{Im} A$, then

$$m \|v\|^2 \leqslant B(v,v) = \langle Av, v \rangle = 0,$$

and we must have $v = 0$. Thus the image of A is dense in H.

Let $w \in \overline{\operatorname{Im} A}$. Then there exists a sequence $\langle w_n \rangle_{n=1}^{\infty}$ of vectors in $\operatorname{Im} A$ converging to w. For each $n \in \mathbb{Z}_+$, there exists a unique vector $u_n \in H$ with $Au_n = w_n$. By the lower bound for B, we have, for all positive integers k and ℓ,

$$\begin{aligned} m \|u_k - u_\ell\|^2 &\leqslant B(u_k - u_\ell, u_k - u_\ell) \\ &= \langle A(u_k - u_\ell), u_k - u_\ell) \rangle \leqslant \|A(u_k - u_\ell)\| \|u_k - u_\ell\|. \end{aligned}$$

This implies that $\|u_k - u_\ell\| \leqslant \frac{1}{m} \|w_k - w_\ell\|$, and since $\langle w_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence, $\langle u_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence as well, converging to some $u \in H$. Finally, by the continuity of A, Au can only be w, and A is surjective.

4. Prove now the *Lax–Milgram theorem*: If B is as above and $\lambda\colon H \longrightarrow B$ is a bounded linear functional, then there exists a unique element $u \in H$ such that for all $v \in H$ we have

$$B(u,v) = \lambda(v).$$

Solution. **Uniqueness.** If u and u' are vectors in H such that

$$B(u,v) = \lambda(v) \quad \text{and} \quad B(u',v) = \lambda(v)$$

for all $v \in H$, then

$$\begin{aligned} m\left\|u-u'\right\|^2 \leqslant B(u-u', u-u') &= B(u, u-u') - B(u', u-u') \\ &= \lambda(u-u') - \lambda(u-u') = 0, \end{aligned}$$

so that $u = u'$.

Existence. By Riesz's representation theorem, there exists a unique $w \in H$ so that $\lambda(v) = \langle w, v\rangle$ for all $v \in H$. If we choose $u = A^{-1}w$, we have

$$B(u,v) = \langle Au, v\rangle = \left\langle AA^{-1}w, v\right\rangle = \langle w, v\rangle = \lambda(v)$$

for all $v \in H$.

Let now $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider the linear partial differential operator

$$L = -\Delta + \sum_{k=1}^{n} b_k(x)\frac{\partial}{\partial x_k} + c(x)$$

where the real valued functions b_k and c are continuous in $\overline{\Omega}$.

5. Define the bilinear form

$$B(u,v) = \int_\Omega \langle \nabla u, \nabla v\rangle + \int_\Omega \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k} v + \int_\Omega cuv$$

on $H_0^1(\Omega) \times H_0^1(\Omega)$. Prove that B satisfies the so-called energy estimates: there exist positive constants M, m and C such that

$$|B(u,v)| \leqslant M\left\|u\right\|_{H_0^1(\Omega)}\left\|v\right\|_{H_0^1(\Omega)}$$

and

$$m\left\|u\right\|_{H_0^1(\Omega)}^2 \leqslant B(u,u) + C\left\|u\right\|_{L^2(\Omega)}^2$$

for all $u, v \in H_0^1(\Omega)$.

Solution. For simplicity, we write $\|\cdot\|$ for the $L^2(\Omega)$-norm, and $\left\|\nabla u\right\|^2$ for $\sum_{k=1}^n \left\|\partial_k u\right\|^2$. By the triangle inequality,

$$|B(u,v)| \leqslant \left\|\nabla u\right\|\left\|\nabla v\right\| + b\left\|\nabla u\right\|\left\|v\right\| + b'\left\|u\right\|\left\|v\right\|,$$

where $b = \max_{1\leqslant k\leqslant n}\left\|b_k\right\|_{L^\infty(\Omega)}$ and $b' = \left\|c\right\|_{L^\infty(\Omega)}$, and so

$$|B(u,v)| \leqslant (1+b+b')\left(\|u\| + \|\nabla u\|\right)\left(\|v\| + \|\nabla v\|\right).$$

By the Cauchy–Schwarz inequality in $\mathbb{R}^2$, we can estimate

$$\|u\| + \|\nabla u\| \leqslant \sqrt{2}\sqrt{\|u\|^2 + \|\nabla u\|^2} = \sqrt{2}\,\|u\|_{H_0^1(\Omega)}\,,$$

and combining this with the previous estimate gives

$$|B(u,v)| \leqslant 2\,(1 + b + b')\,\|u\|_{H_0^1(\Omega)}\,\|v\|_{H_0^1(\Omega)}\,,$$

which is an upper bound of the desired shape.

Again, by the triangle inequality, we have

$$\begin{aligned} B(u,u) &= \int_\Omega |\nabla u|^2 + \sum_{k=1}^n \int_\Omega b_k \partial_k u \cdot u + \int_\Omega c\,|u|^2 \\ &\geqslant \|\nabla u\|^2 - b\,\|\nabla u\|\,\|u\| - b'\,\|u\|^2\,. \end{aligned}$$

The elementary inequality $\alpha\beta \leqslant \frac{\alpha^2+\beta^2}{2}$, which holds for all $\alpha, \beta \in [0,\infty[$, implies that

$$b\,\|\nabla u\|\,\|u\| = \|\nabla u\| \cdot b\,\|u\| \leqslant \frac{1}{2}\,\|\nabla u\|^2 + \frac{b^2}{2}\,\|u\|^2\,.$$

Combining this with the lower bound for $B(u,u)$ gives

$$B(u,u) \geqslant \frac{1}{2}\,\|\nabla u\|^2 - \left(\frac{b^2}{2} + b'\right)\|u\|^2 = \frac{1}{2}\,\|u\|^2_{H_0^1(\Omega)} - \left(\frac{b^2}{2} + b' + \frac{1}{2}\right)\|u\|^2\,,$$

and we are done.

6. Apply the previous exercise to study the weak solvability on $H_0^1(\Omega)$ of the boundary value problem

$$Lu + \mu u = f \text{ in } \Omega,\ u|_{\partial\Omega} = 0$$

for a large enough constant μ.

Solution. Here weak solvability means that $u \in H_0^1(\Omega)$ is such that

$$B(u,v) + \mu \int_\Omega uv = \int_\Omega fv$$

for all $v \in H_0^1(\Omega)$. We assume that $f \in L^2(\Omega)$. Write $\widetilde{B}(u,v)$ for the left-hand side. From the previous exercise, we know that

$$\left|\widetilde{B}(u,v)\right| \leqslant (2 + 2b + 2b' + |\mu|)\,\|u\|_{H_0^1(\Omega)}\,\|v\|_{H_0^1(\Omega)}\,,$$

and if $\mu > \frac{b^2}{2} + b' + \frac{1}{2}$, then

$$\begin{aligned} \widetilde{B}(u,u) &\geqslant \frac{1}{2}\,\|u\|_{H_0^1(\Omega)} + \left(\mu - \frac{b^2}{2} - b' - 1\right)\|u\|^2 \\ &\geqslant \min\left\{\frac{1}{2}, \mu - \frac{b^2}{2} - b' - \frac{1}{2}\right\}\|u\|^2_{H_0^1(\Omega)}\,. \end{aligned}$$

Also, the mapping $\lambda = v \longmapsto \int_\Omega fv$ is a bounded linear functional of $H_0^1(\Omega)$, as

$$|\lambda(v)| \leqslant \|f\|\,\|v\| \leqslant \|f\|\,\|v\|_{H_0^1(\Omega)}\,.$$

Thus, by the Lax–Milgram theorem, there is a unique weak solution $u \in H_0^1(\Omega)$.

7. Show that the set of Dirichlet eigenvalues of Δ on $\Omega \subset \mathbb{R}^n$ is invariant under rotations, reflections and translations of Ω.

Solution. The key point here is that the Laplace operator commutes with the mappings in question, and more generally with automorphisms of the Euclidean space (as a geometrical structure). In other words, for such a geometrical mapping $A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and for $f \in C^2(\mathbb{R}^n)$,

$$\Delta\big(f(A(x))\big) = \big(\Delta f\big)(A(x)), \tag{$*$}$$

$x \in \mathbb{R}^n$. As the group of automorphisms in question is generated by translations and orthogonal transformations, it is enough to prove $(*)$ for those two classes of mappings. For translations $(*)$ is clearly true, so we may focus on the latter class.

Let $O = [O_{ij}] \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, i.e. $O^T O = I$. In terms of the components, orthogonality means that

$$\sum_{k=1}^{n} O_{ik} O_{jk} = \delta_{ij},$$

where $\delta_{ij} = 1$ when $i = j$ and $= 0$ otherwise. Given a vector $x \in \mathbb{R}^n$, the kth component $(Ox)_k$ of Ox is

$$(Ox)_k = \sum_{j=1}^{n} O_{kj} x_j.$$

Now, using the above relations and the chain rule,

$$\begin{aligned}
\Delta\big(f(Ox)\big) &= \sum_{\ell=1}^{n} \frac{\partial^2}{\partial x_\ell^2}\big(f(Ox)\big) = \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell} \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(Ox) \cdot \frac{\partial (Ox)_k}{\partial x_\ell} \\
&= \sum_{\ell=1}^{n} \sum_{k=1}^{n} \sum_{k'=1}^{n} \frac{\partial^2 f}{\partial x_{k'} \partial x_k}(Ox) \cdot \frac{\partial (Ox)_{k'}}{\partial x_\ell} \cdot O_{k\ell} \\
&= \sum_{k=1}^{n} \sum_{k'=1}^{n} \frac{\partial^2 f}{\partial x_{k'} \partial x_k}(Ox) \sum_{\ell=1}^{n} O_{k'\ell} O_{k\ell} = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}(Ox) = \big(\Delta f\big)(Ox).
\end{aligned}$$

Now that $(*)$ has been proved, let u be a Dirichlet eigenfunction of $-\Delta$ in Ω corresponding to an eigenvalue λ. Then

$$-\Delta\big(u(A^{-1}x)\big)\big) = -\big(\Delta u\big)(A^{-1}\cdot) = \lambda u(A^{-1}\cdot),$$

so that $u(A^{-1}\cdot)$ is a Dirichlet eigenfunction of $-\Delta$ in $A[\Omega]$ corresponding to the eigenvalue λ.

8. Given $\lambda > 0$ and $\Omega \subset \mathbb{R}^d$, let $\lambda\Omega = \{\lambda x | x \in \Omega\}$. What can you say about the Dirichlet eigenvalues of $\lambda\Omega$?

Solution. Let $u \in C^2_\partial(\Omega)$ be a Dirichlet eigenfunction of $-\Delta$ in Ω corresponding to an eigenvalue μ. Then $u(\cdot/\lambda)$ is a function in $C^2_\partial(\lambda\Omega)$ and

$$-\Delta\left(u\Big(\frac{\cdot}{\lambda}\Big)\right) = -\frac{1}{\lambda^2}(\Delta u)\Big(\frac{\cdot}{\lambda}\Big) = \frac{\mu}{\lambda^2}u\Big(\frac{\cdot}{\lambda}\Big),$$

so that $u(\cdot/\lambda)$ is a Dirichlet eigenfunction of $-\Delta$ in $\lambda\Omega$ corresponding to the eigenvalue μ/λ^2.

Applying the same argument with the inverse of λ shows, that if μ' is a Dirichlet eigenvalue of $-\Delta$ in $\lambda\Omega$, then $\lambda^2\mu'$ is a Dirichlet eigenvalue of $-\Delta$ in Ω.

For the next two exercises fix a bounded domain $\Omega \subset \mathbb{R}^d$, let

$$C^2_\partial(\Omega) = \left\{u \in C^2(\Omega) \cap C(\overline{\Omega}) \big| u|_{\partial\Omega} = 0\right\}$$

and define

$$\lambda_1 = \inf_{w \in C^2_\partial(\Omega)} \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}}.$$

9. Assume $u \in C^2_\partial(\Omega)$ is such that

$$\lambda_1 = \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}},$$

i.e. we attain the minimum at u. Prove that λ_1 is a Dirichlet eigenvalue of $-\Delta$ on Ω with eigenvalue u. **Hint:** Given any $v \in C^2_\partial(\Omega)$ study the function

$$f(\varepsilon) = \frac{\|\nabla(u+\varepsilon v)\|^2_{L^2(\Omega)}}{\|u+\varepsilon v\|^2_{L^2(\Omega)}},$$

at zero.

Solution. Again, for simplicity, we denote the L^2-norm in Ω by $\|\cdot\|$, and the inner product by $\langle\cdot|\cdot\rangle$. Let us first compute the derivative $f'(\varepsilon)$:

$$\begin{aligned} f'(\varepsilon) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\|\nabla u\|^2 + 2\varepsilon\langle\nabla u|\nabla v\rangle + \varepsilon^2\|\nabla v\|^2}{\|u\|^2 + 2\varepsilon\langle u|v\rangle + \varepsilon^2\|v\|^2} \\ &= \frac{2\langle\nabla u|\nabla v\rangle + 2\varepsilon\|\nabla v\|^2}{\|u\|^2 + 2\varepsilon\langle u|v\rangle + \varepsilon^2\|v\|^2} \\ &\quad - \frac{\left(\|\nabla u\|^2 + 2\varepsilon\langle\nabla u|\nabla v\rangle + \varepsilon^2\|\nabla v\|^2\right)\left(2\langle u|v\rangle + 2\varepsilon\|v\|^2\right)}{\left(\|u\|^2 + 2\varepsilon\langle u|v\rangle + \varepsilon^2\|v\|^2\right)^2}. \end{aligned}$$

Since u is a minimum, $f(\varepsilon)$ has a minimum at $\varepsilon = 0$, and we must have $f'(0) = 0$. More precisely,

$$\frac{2\langle\nabla u|\nabla v\rangle}{\|u\|^2} - \frac{\left(\|\nabla u\|^2\right)\left(2\langle u|v\rangle\right)}{\left(\|u\|^2\right)^2} = 0,$$

for all v in, say, $C_c^\infty(\Omega)$. This simplifies to

$$\langle \nabla u | \nabla v \rangle = \frac{\|\nabla u\|^2}{\|u\|^2} \langle u | v \rangle = \lambda_1 \langle u | v \rangle .$$

By Green's formulae, we have

$$\langle -\Delta u | v \rangle = \lambda_1 \langle u | v \rangle$$

for all test functions v. Since test functions are dense in $L^2(\Omega)$, we conclude that $-\Delta u = \lambda_1 u$.

10. Prove that $\lambda_1 \leqslant \lambda$ for all Dirichlet eigenvalues λ of $-\Delta$ on Ω.

Solution. If $u \in C_\partial^2(\Omega)$ solves $-\Delta u = \lambda u$, where $\lambda \in \mathbb{R}$, then

$$-\int_\Omega u \Delta u = \lambda \int_\Omega |u|^2 .$$

By Green's formulae, we have

$$\int_\Omega |\nabla u|^2 = \lambda \int_\Omega |u|^2 .$$

Thus, directly by the definition of λ_1, we have

$$\lambda_1 \leqslant \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} = \lambda.$$