Integral equations
Solutions to the second problem set

1. Give a detailed proof for the convergence of the series defining the resolvent kernel of a Volterra equation of the second kind with a weakly singular kernel.

Solution. So we need to prove that the series

$$\sum_{n=n_0}^{\infty} \lambda^{n-1} K^{(n)}(s,t)$$

converges absolutely and uniformly. This will follow from the estimate

$$\left|K^{(n)}(s,t)\right| \leqslant \frac{C^n\,\Gamma^n(1-\alpha)}{\Gamma\big(n(1-\alpha)\big)}\,|s-t|^{n(1-\alpha)-1},$$

which holds for all s and t and for each $n \in \mathbb{Z}_+$, and where $C \in \mathbb{R}_+$ is such that

$$|K(s,t)| \leqslant \frac{C}{|s-t|^{\alpha}}.$$

We shall prove this by induction on n. The case $n=1$ is certainly true, so assume we know the estimate for some $K^{(n)}$, and let us consider $K^{(n+1)}$. By mimicking the computations done in the section on weakly singular kernels in the lecture notes, we get, for $t \leqslant s$,

$$\begin{aligned}
\left|K^{(n+1)}(s,t)\right| &= \left|\int_t^s K^{(n)}(s,r)\,K(r,t)\,\mathrm{d}r\right| \\
&\leqslant \frac{C^{n+1}\,\Gamma^n(1-\alpha)}{\Gamma\big(n(1-\alpha)\big)} \int_t^s \frac{(s-r)^{n(1-\alpha)-1}\,\mathrm{d}r}{(r-t)^{\alpha}} \\
&= \frac{C^{n+1}\,\Gamma^n(1-\alpha)}{\Gamma\big(n(1-\alpha)\big)}\,(s-t)^{(n+1)(1-\alpha)-1} \int_0^1 (1-w)^{n(1-\alpha)-1}\,w^{1-\alpha-1}\,\mathrm{d}w,
\end{aligned}$$

and the desired estimate follows since

$$\int_0^1 (1-w)^{n(1-\alpha)-1}\,w^{1-\alpha-1}\,\mathrm{d}w = \mathrm{B}\big(n(1-\alpha),1-\alpha\big) = \frac{\Gamma\big(n(1-\alpha)\big)\,\Gamma(1-\alpha)}{\Gamma\big((n+1)(1-\alpha)\big)}.$$

The estimate for $s \leqslant t$ is similar.

2. Consider the example from mechanics in Section 1.6 of lecture notes: find the solution in the case when $f(x) = T$, i.e. when a particle is released from height $x > 0$, it always takes a constant time $T > 0$ to travel along the curve $y = F(x)$ to zero height. Find the equation of F, or at least a series approximation to it.

Solution. So, our task here is to derive a reasonable equation for $F(x)$ from the integral equation

$$\int_0^x \frac{\sqrt{1+(F'(t))^2}\,\mathrm{d}t}{\sqrt{2g(x-t)}} = T.$$

We shall use the work done in the lecture notes with the notation

$$f(x) = \sqrt{2g}\,T, \quad G(s,t) = 1, \quad \alpha = \frac{1}{2}, \quad \text{and} \quad \varphi(t) = \sqrt{1+(F'(t))^2}.$$

Then φ satisfies the equation

$$\int_0^x K_1(x,t)\,\varphi(t)\,\mathrm{d}t = f_1(x).$$

Here, using the substitution $t = x\cos^2 u$,

$$f_1(x) = \int_0^x \frac{f(t)\,\mathrm{d}t}{(x-t)^{1/2}} = \sqrt{2g}\,T \int_0^x \frac{\mathrm{d}t}{(x-t)^{1/2}} = \sqrt{2g}\,T\sqrt{x}\int_0^{\pi/2} \frac{2\sin u\,\cos u\,\mathrm{d}u}{\sin u}$$

$$= 2\sqrt{2g}\,T\sqrt{x}\int_0^{\pi/2}\cos u\,\mathrm{d}u = 2\sqrt{2g}\,T\sqrt{x}\,\sin u\Big]_0^{u=\pi/2} = 2\sqrt{2g}\,T\sqrt{x}.$$

In the same vein,

$$K_1(x,t) = \int_0^1 \frac{G(t+r(x-t),t)\,\mathrm{d}r}{(1-r)^{1/2}\,r^{1/2}} = \int_0^1 (1-r)^{-1/2}\,r^{-1/2}\,\mathrm{d}r = \frac{\pi}{\sin\frac{\pi}{2}} = \pi.$$

Thus, φ solves the equation

$$\int_0^x \varphi(t)\,\mathrm{d}t = \frac{2\sqrt{2g}\,T\sqrt{x}}{\pi}.$$

Differentiating this gives

$$\sqrt{1+(F'(x))^2} = \varphi(x) = \frac{\sqrt{2g}\,T}{\pi\sqrt{x}},$$

and squaring gives

$$(F'(x))^2 = \frac{2g\,T^2}{\pi^2\,x} - 1,$$

which is a differential equation for F. The curves $y = F(x)$ which arise from this differential equation turn out to be cycloids.

3. Consider a **nonlinear** Volterra equation of the second kind,

$$\varphi(s) + \int_0^s K\big(s,t,\varphi(t)\big)\,\mathrm{d}t = f(s). \qquad (*)$$

Assume the following: the function $K(x,y,z)$ is continuous in the set D defined by

$$|x|\,,|y| \leqslant a, \qquad |z| \leqslant b,$$

and that K is uniformly Lipschitz continuous in z,

$$|K(x,y,z_1) - K(x,y,z_2)| \leqslant K\,|z_1 - z_2|\,, \qquad \langle x,y,z_i\rangle \in D.$$

Also, assume that $f \in C\big([-a,a]\big)$, $f(0) = 0$, and that f satisfies the Lipschitz condition

$$|f(x_1) - f(x_2)| \leqslant k\,|x_1 - x_2|\,, \qquad |x_i| \leqslant a.$$

Let

$$M = \sup_D |K|\,.$$

Show that the iteration

$$\varphi_0(s) = f(s), \qquad \varphi_n(s) = f(s) - \int_0^s K\big(s,t,\varphi_{n-1}(t)\big)\,\mathrm{d}t$$

converges in the set

$$|s| \leqslant a', \qquad a' = \min\left\{a, \frac{b}{k+M}\right\},$$

and that the limit is a solution of $(*)$ on the interval $[-a',a']$.

Solution. We first have to check that each φ_n takes values only in the interval $[-b,b]$, because its values will be fed into K. First, for $s \in [-a',a']$, the Lipschitz property of f implies that

$$|\varphi_0(s)| = |f(s) - f(0)| \leqslant k\,|s - 0| \leqslant k\,a' \leqslant \frac{k\,b}{k+M} \leqslant b.$$

Next, assume that $-b \leqslant \varphi_n(s) \leqslant b$ for all $s \in [-a',a']$ for some $n \in \mathbb{Z}_+ \cup \{0\}$. Then we may estimate for $s \in [-a',a']$ that

$$|\varphi_{n+1}(s)| \leqslant |f(s)| + \left|\int_0^s \big|K\big(s,t,\varphi_n(t)\big)\big|\,\mathrm{d}t\right| \leqslant k\,a' + M\,a' \leqslant \frac{k\,b}{k+M} + \frac{M\,b}{k+M} = b.$$

Thus the sequence $\langle\varphi_n\rangle_{n=0}^{\infty}$ is a well-defined sequence of continuous functions defined in $[-a',a']$ and taking values in $[-b,b]$.

We shall prove by induction on n that

$$|\varphi_n(s) - \varphi_{n-1}(s)| \leqslant \frac{M\,K^{n-1}\,|s|^n}{n!}$$

for all $s \in [-a',a']$ and for each $n \in \mathbb{Z}_+$. First we observe that

$$|\varphi_1(s) - \varphi_0(s)| = \left|\int_0^s K\big(s,t,\varphi_0(t)\big)\,\mathrm{d}t\right| \leqslant |s|\,M.$$

Next, assuming that we have proved the inequality for some $n \in \mathbb{Z}_+$, we estimate

$$|\varphi_{n+1}(s) - \varphi_n(s)| \leqslant \left|\int_0^s \big|K\big(s,t,\varphi_n(t)\big) - K\big(s,t,\varphi_{n-1}(t)\big)\big|\,\mathrm{d}t\right|$$

$$\leqslant K\left|\int_0^s |\varphi_n(t) - \varphi_{n-1}(t)|\,\mathrm{d}t\right| \leqslant K\left|\int_0^s \frac{M\,K^{n-1}\,|s|^n}{n!}\mathrm{d}t\right| = \frac{M\,K^n\,|s|^{n+1}}{(n+1)!}.$$

Now the series

$$\varphi_0 + (\varphi_1 - \varphi_0) + (\varphi_2 - \varphi_1) + \dots$$

converges absolutely and uniformly in $[-a', a']$, and so we know that the sequence $\langle \varphi_n \rangle_{n=0}^{\infty}$ converges uniformly in the interval $[-a', a']$ to a continuous function, which we shall call φ. We note that φ can only take values in the interval $[-b, b]$ as each of the functions φ_n does.

We also have

$$\int_0^s K\big(s, t, \varphi_n(t)\big)\,\mathrm{d}t \longrightarrow \int_0^s K\big(s, t, \varphi(t)\big)\,\mathrm{d}t$$

uniformly in $s \in [-a', a']$ as $n \longrightarrow \infty$. This follows from the estimates

$$\left| \int_0^s K\big(s, t, \varphi_n(t)\big)\,\mathrm{d}t - \int_0^s K\big(s, t, \varphi(t)\big)\,\mathrm{d}t \right| \leqslant K\,|s| \max_{|t| \leqslant |s|} |\varphi_n(t) - \varphi(t)|\,,$$

and the fact that $\varphi_n(t)$ tends uniformly to $\varphi(t)$.

Thus, in the limit $n \longrightarrow \infty$, the equation which defined φ_n in terms of φ_{n-1} becomes

$$\varphi(s) = f(s) - \int_0^s K\big(s, t, \varphi(t)\big)\,\mathrm{d}t,$$

and so φ is indeed a solution.

An alternative approach. Let us also show another way of approaching this kind of a problem. The following solution is not strictly speaking a solution unless $a'\,K < 1$, or unless we set

$$a' = \min\left\{ a, \frac{b}{k+M}, \frac{1}{K+1} \right\},$$

or so, but it is nonetheless worth giving here. Let us first recall the following important fact from the topology of metric spaces:

The contraction principle. *Let X be a closed metric space with metric d, and let $A\colon X \longrightarrow X$ be a* ***contraction****, i.e. assume that there exists a constant $c \in \,]0,1[$ such that*

$$d\big(A(x), A(y)\big) \leqslant c\, d(x, y)$$

for all $x, y \in X$. Then there exists a unique point $y \in X$ such that $A(y) = y$, and furthermore, given any point $x \in X$, the sequence

$$x, \quad A(x), \quad A(A(x)), \quad A(A(A(x))), \quad \dots$$

converges to y.

Proof. Let $y, z \in X$ be such that $A(y) = y$ and $A(z) = z$. Then

$$d(y,z) = d\big(A(y), A(z)\big) \leqslant c\, d(y,z),$$

which is only possible if $d(y,z) = 0$. Thus the fixed point y, if it exists, must be unique.

Next, let $x \in X$ be arbitrary, and let us consider the sequence

$$y_0 = x, \quad y_1 = A(x), \quad y_2 = A(A(x)), \quad y_3 = A(A(A(x))), \quad \ldots$$

If the sequence $\langle y_n \rangle_{n=0}^{\infty}$ is Cauchy in X, then it converges to some $y \in X$, and in view of the manifest continuity of A, the relation $y_n = A(y_{n-1})$, which holds for all $n \in \mathbb{Z}_+$, becomes $y = A(y)$ in the limit $n \longrightarrow \infty$, thereby establishing the existence of a fixed point.

Therefore, we only have to prove that $\langle y_n \rangle_{n=0}^{\infty}$ is Cauchy. To see this, let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+$. Then

$$\begin{aligned} d(y_n, y_{n+k}) &\leqslant d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+k-1}, y_{n+k}) \\ &\leqslant c^n\, d(y_0, y_1) + c^{n+1}\, d(y_0, y_1) + \ldots + c^{n+k-1}\, d(y_0, y_1) \\ &\leqslant \big(c^n + c^{n+1} + \ldots\big)\, d(y_0, y_1) = \frac{c^n}{1-c}\, d(y_0, y_1), \end{aligned}$$

and since the last expression is independent of k and tends to zero as $n \longrightarrow \infty$, we are done.

Now, the idea of the solution is to apply the contraction principle to the space

$$X = \big\{\varphi \in C\big([-a', a']\big) \big| \varphi\big[[-a', a']\big] \subseteq [-b, b]\big\},$$

which we shall equip with the metric $d(\varphi, \psi) = \|\varphi - \psi\|_\infty$, defined for $\varphi, \psi \in X$. A Cauchy sequence $\langle \varphi_n \rangle_{n=1}^{\infty}$ in X is a Cauchy sequence in the Banach space $C\big([-a', a']\big)$, and therefore converges to some $\varphi \in C\big([-a', a']\big)$. Since $|\varphi_n(x)| \leqslant b$ for all $s \in [-a', a']$ and for each $n \in \mathbb{Z}_+$, we clearly must also have $|\varphi(s)| \leqslant b$ for all $s \in [-a', a']$, so that $\varphi \in X$ and X is complete.

Let us next observe that $f \in X$. We know that f is continuous in $[-a', a']$, so we only have to check that the image of f is contained in $[-b, b]$. This is so because, for $s \in [-a', a']$, the Lipschitz property of f implies that

$$|f(s)| = |f(s) - f(0)| \leqslant k\,|s - 0| \leqslant k\,a' \leqslant \frac{k\,b}{k+M} \leqslant b.$$

We shall define for $\varphi \in X$ an operator A by the formula

$$\big(A\varphi\big)(s) = f(s) - \int_0^s K\big(s, t, \varphi(t)\big)\, \mathrm{d}t$$

for all $s \in [-a', a']$. At first we only know that A maps X into $C\big([-a', a']\big)$, but, for $s \in [-a', a']$, we may estimate

$$\big|\big(A\varphi\big)(s)\big| \leqslant |f(s)| + \left|\int_0^s \big|K\big(s, t, \varphi(t)\big)\big|\, \mathrm{d}t\right| \leqslant k\,a' + M\,a' \leqslant \frac{k\,b}{k+M} + \frac{M\,b}{k+M} = b,$$

and so we have $A\varphi \in X$ and $A\colon X \longrightarrow X$.

Finally, we only have to prove that A is a contraction. Let $\varphi, \psi \in X$ be arbitrary. Then, for all $s \in [-a', a']$, we have

$$\begin{aligned} \big|(A\varphi)(s) - (A\psi)(s)\big| &\leqslant \left| \int_0^s \big|K\big(s,t,\psi(t)\big) - K\big(s,t,\varphi(t)\big)\big| \,\mathrm{d}t \right| \\ &\leqslant \left| \int_0^s K\,|\psi(t) - \varphi(t)| \,\mathrm{d}t \right| \leqslant a'\,K\,d(\psi,\varphi), \end{aligned}$$

and so

$$d(A\varphi, A\psi) \leqslant a'\,K\,d(\varphi,\psi),$$

and A is a contraction, given that $a'K < 1$.

4. Let $\langle X, \langle\cdot|\cdot\rangle\rangle$ be an inner product space and $\|\cdot\|$ the induced norm. Prove that an inner product satisfies the so called parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2\,, \qquad x,y \in X.$$

Solution. We write the norms in terms of the inner product:

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \langle x+y|x+y\rangle + \langle x-y|x-y\rangle \\ &= \langle x|x\rangle + \langle x|y\rangle + \langle y|x\rangle + \langle y|y\rangle \\ &\quad + \langle x|x\rangle - \langle x|y\rangle - \langle y|x\rangle + \langle y|y\rangle \\ &= 2\,\langle x|x\rangle + 2\,\langle y|y\rangle = 2\,\|x\|^2 + 2\,\|y\|^2\,. \end{aligned}$$

5. Consider the space $C\big([a,b]\big)$. Show that the sup-norm

$$\|f\|_{\sup} = \sup_{x\in[a,b]} |f(x)|$$

is not determined by any inner product.

Solution. The idea is to pick some functions $f, g \in C\big([a,b]\big)$ for which the parallelogram identity is not satisfied. One choice could be to choose the function f so that $\|f\|_{\sup} = 1$ and that f vanishes in $\big[\frac{a+b}{2}, b\big]$, and the function g so that $\|g\|_{\sup} = 1$ and that g vanishes in $\big[a, \frac{a+b}{2}\big]$. Then

$$\|f+g\|_{\sup} = \|f-g\|_{\sup} = \|f\|_{\sup} = \|g\|_{\sup} = 1,$$

and the parallelogram identity clearly can not hold as $1+1 \neq 2+2$.

6. Similarly, consider the L^p-norms

$$\|f\|_p = \left(\int_a^b |f(x)|^p \,\mathrm{d}x \right)^{1/p},$$

where $1 \leqslant p < \infty$. Prove that if $p \neq 2$ then this norm is not induced by any inner product.

Solution. We proceed as in the previous solution: we pick two functions $f, g \in L^2([a,b])$ for which the parallelogram identity fails. We write $\ell = \frac{b-a}{2}$, and choose f to be 1 in $[a, a+\ell[$, and 0 in $[b-\ell, b]$. We also choose $g = 1 - f$. Now

$$\|f+g\|^{2/p} + \|f-g\|^{2/p} = 2(2\ell)^{2/p} = 2 \cdot 2^{2/p}\ell^{2/p},$$

whereas

$$2\|f\|^{2/p} + 2\|g\|^{2/p} = 4\ell^{2/p},$$

and $2 \cdot 2^{2/p} \neq 4$ unless $p = 2$.

7. Let $\langle X_i, \|\cdot\| \rangle$ be normed spaces, $i = 1, 2, 3$. Show that, for the norm of a linear operator $A\colon X_1 \longrightarrow X_2$, we have

$$\|A\| = \sup_{0<\|x\|_1 \leqslant 1} \frac{\|Ax\|_2}{\|x\|_1} = \sup_{\|x\|_1 = 1} \frac{\|Ax\|_2}{\|x\|_1}.$$

Also, let $B\colon X_2 \longrightarrow X_3$ be linear. Prove that

$$\|BA\| \leqslant \|B\| \cdot \|A\|.$$

Solution. The first assertion follows from the fact that, for any $x \in X_1$ with $0 < \|x\|_1 \leqslant 1$,

$$\frac{\|Ax\|_2}{\|x\|} = \frac{\left\|A\frac{x}{\|x\|_1}\right\|_2}{\left\|\frac{x}{\|x\|_1}\right\|_1},$$

and $\left\|\frac{x}{\|x\|_1}\right\|_1 = 1$, so that the supremum over those x with $0 < \|x\|_1 \leqslant 1$ is really over the same set of values as the supremum over x satisfying $\|x\|_1 = 1$.

The second assertion follows from the estimates

$$\|BAx\| \leqslant \|B\| \cdot \|Ax\| \leqslant \|B\| \cdot \|A\| \cdot \|x\|,$$

which hold for all $x \in X_1$.

8. Consider the integral equation

$$f(x) + \frac{1}{20}\int_0^1 e^{-|xy|^2} \sin(x^2+y^2)\, f(y)\,\mathrm{d}y = \sin x.$$

Prove that this has a unique solution $L^2([0,1])$, and that in fact this solution is also continuous.

Solution. Let K be the linear operator $L^2([0,1]) \longrightarrow L^2([0,1])$ defined for a given $f \in L^2([0,1])$ by the formula

$$(Kf)(x) = \frac{1}{20}\int_0^1 e^{-|xy|^2} \sin\left(x^2+y^2\right) f(y)\,\mathrm{d}y$$

for all $x \in [0,1]$. This is an integral operator with the kernel function

$$K(x,y) = \frac{1}{20}e^{-|xy|^2} \sin\left(x^2+y^2\right).$$

Since clearly

$$\sup_{x\in[0,1]} \int_0^1 |K(x,y)|\,\mathrm{d}y \leqslant \frac{1}{20} \quad \text{and} \quad \sup_{y\in[0,1]} \int_0^1 |K(x,y)\|\,\mathrm{d}x \leqslant \frac{1}{20},$$

Schur's lemma tells us that K is a bounded linear operator of $L^2\big([0,1]\big)$ with operator norm $\|K\| \leqslant \frac{1}{20}$. Thus, the operator $I - K$ is invertible by Neumann series and the equation

$$f - Kf = g$$

has a unique solution in $L^2\big([0,1]\big)$ for any given $g \in L^2\big([0,1]\big)$.

To satisfy the second demand of the exercise, we will prove that Kf is continuous for any $f \in L^2\big([0,1]\big)$. For this purpose, let $x, x' \in [0,1]$ be arbitrary. Since K is infinitely smooth, the supremum

$$M = \sup_{\substack{x\in[0,1],\\ y\in[0,1]}} \big|\big(\partial_1 K\big)(x,y)\big|$$

is finite. Now the continuity of Kf follows from the estimates

$$\begin{aligned}
\big|\big(Kf\big)(x) - \big(Kf\big)(x')\big| &= \left|\int_0^1 \big(K(x,y) - K(x',y)\big)\, f(y)\,\mathrm{d}y\right| \\
&\leqslant \int_0^1 \left|\int_{x'}^{x} \big(\partial_1 K\big)(\xi,y)\,\mathrm{d}\xi\right| \cdot |f(y)|\,\mathrm{d}y \\
&\leqslant \sqrt{\int_0^1 \left|\int_{x'}^{x} \big(\partial_1 K\big)(\xi,y)\,\mathrm{d}\xi\right|^2 \mathrm{d}y}\; \|f\|_{L^2} \leqslant M\,|x - x'|\, \|f\|_{L^2}\,.
\end{aligned}$$

9. Let H be a Hilbert space, and $A\colon H \longrightarrow H$ a bounded linear map for which $\|A^{n_0}\| < 1$ for some positive integer n_0. Prove that $I - A$ is invertible and determine its inverse.

Solution. Let $n \in \mathbb{Z}_+$ be arbitrary and divide it by n_0 to get a representation $n = qn_0 + r$ with $q \in \mathbb{Z}_+ \cup \{0\}$ and $r \in \{0, 1, \dots, n_0 - 1\}$. Then, writing

$$C = \max\big\{1, \big\|A\big\|, \big\|A^2\big\|, \dots, \big\|A^{n_0-1}\big\|\big\},$$

we may estimate

$$\|A^n\| \leqslant \|A^{n_0}\|^q\, \|A^r\| \leqslant C\, \|A^{n_0}\|^q\,.$$

By this and the triangle inequality we may estimate

$$\left\|\sum_{n=0}^{\infty} A^n\right\| \leqslant \sum_{n=0}^{\infty} \|A^n\| \leqslant C\, n_0 \sum_{q=0}^{\infty} \|A^{n_0}\|^q < \infty,$$

so that the series $1 + A + A^2 + \dots$ converges absolutely to a bounded operator $S\colon H \longrightarrow H$. Also, we have $\|A^n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Let us observe next that

$$(1 - A)\, S = \lim_{N \longrightarrow \infty} (1 - A) \sum_{n=0}^{N} A^n = \lim_{N \longrightarrow \infty} \left(1 - A^{N+1}\right) = 1,$$

as well as

$$S\, (1 - A) = \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} A^n\, (1 - A) = \lim_{N \longrightarrow \infty} \left(1 - A^{N+1}\right) = 1,$$

and therefore $1 - A$ is invertible with inverse S.