Integral equations
Solutions to the first problem set

Before looking at the problems, let us recall two useful facts from the lectures:

Lemma. *Let F be a continuous function from $I \times I$ to $\mathbb{R}$, where I is an open interval of $\mathbb{R}$ containing zero, and assume that F is continuously differentiable with respect to the first variable. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_0^s F(s,t)\,\mathrm{d}t = F(s,s) + \int_0^s \frac{\partial}{\partial s} F(s,t)\,\mathrm{d}t.$$

for $s \in I$.

Lemma. Let F be a continuous real-valued function in $[0,x] \times [0,x]$, where $x \in \mathbb{R}$. Then

$$\int_0^x \int_0^t F(t,u)\,\mathrm{d}u\,\mathrm{d}t = \int_0^x \int_u^x F(t,u)\,\mathrm{d}t\,\mathrm{d}u.$$

Proof. This follows easily from Fubini's theorem if we introduce a function $\chi\colon\ [0,x] \times [0,x] \longrightarrow \mathbb{R}$ for $s,t \in [0,x]$ by

$$\chi(t,u) = \begin{cases} 1 & \text{if } t \geqslant u, \\ 0 & \text{if } t < u, \end{cases}$$

for then

$$\begin{aligned}
\int_0^x \int_0^t F(t,u)\,\mathrm{d}u\,\mathrm{d}t &= \int_0^x \left(\int_0^t \chi(t,u)\,F(t,u)\,\mathrm{d}u + \int_t^x \chi(t,u)\,F(t,u)\,\mathrm{d}u \right) \mathrm{d}t \\
&= \int_0^x \int_0^x \chi(t,u)\,F(t,u)\,\mathrm{d}u\,\mathrm{d}t = \int_0^x \int_0^x \chi(t,u)\,F(t,u)\,\mathrm{d}t\,\mathrm{d}u \\
&= \int_0^x \left(\int_0^u \chi(t,u)\,F(t,u)\,\mathrm{d}t + \int_u^x \chi(t,u)\,F(t,u)\,\mathrm{d}t \right) \mathrm{d}u \\
&= \int_0^x \int_u^x F(t,u)\,\mathrm{d}t\,\mathrm{d}u.
\end{aligned}$$

1. Solve the Volterra equation

$$\varphi(s) - \int_0^s (s-t)\,\varphi(t)\,\mathrm{d}t = 2s.$$

Solution. The idea of the solution is to reduce the equation to an initial value problem for a differential equation through repeated differentiation.

The substitution $s = 0$ shows that a solution must satisfy $\varphi(0) = 0$. The equation also implies that φ is continuously differentiable as the other two terms are, assuming that φ is at least, say, continuous. Differentiating the equation gives

$$\varphi'(s) - \int_0^s \varphi(t)\,\mathrm{d}t = 2.$$

Substituting again $s = 0$ gives another initial value condition $\varphi'(0) = 2$, and the equation implies that φ' must also be continuously differentiable. Taking derivatives once more we land into the differential equation

$$\varphi''(s) - \varphi(s) = 0.$$

A solution to this must be of the form

$$\varphi(s) = A\cosh s + B\sinh s$$

for some constants A and B.

Since $0 = \varphi(0) = A$, we have $\varphi(s) = B\sinh s$ for some constant B. Furthermore, since $2 = \varphi'(0) = B$, we conclude that the only possible solution to the original integral equation is

$$\varphi(s) = 2\sinh s.$$

Finally, this really is a solution as we know from the lectures that a solution must exist.

2. Solve the Volterra equation

$$\varphi(s) - 4\int_0^s (s-t)\,\varphi(t)\,\mathrm{d}t = s^3.$$

Solution. We proceed as in the previous solution. The substitution $s = 0$ gives the initial value condition $\varphi(0) = 0$. Assuming that the solution is at least continuous, the integral equation implies that it must be at least once continuously differentiable. Differentiating the integral equation then gives

$$\varphi'(s) - 4\int_0^s \varphi(t)\,\mathrm{d}t = 3s^2.$$

This implies another initial value condition $\varphi'(0) = 0$ and that $\varphi'(s)$ must be continuously differentiable, too. Taking derivatives again gives the differential equation

$$\varphi''(s) - 4\varphi(s) = 6s.$$

Now we have to solve an inhomogeneous initial value problem. One obvious solution to the differential equation is given by $\varphi(s) = -\frac{3s}{2}$. Thus every classical solution to the differential equation is of the form

$$\varphi(s) = A\cosh 2s + B\sinh 2s - \frac{3s}{2}$$

for some constants A and B.

We must have $0 = \varphi(0) = A$ and so a solution to the integral equation must be of the form

$$\varphi(s) = B \sinh 2s - \frac{3s}{2}.$$

Since $\varphi'(s) = 2B \cosh 2s - \frac{3}{2}$, the initial value condition $\varphi'(0) = 0$ implies that $2B = \frac{3}{2}$. Thus the only possible solution is

$$\varphi(s) = \frac{3}{4} \sinh 2s - \frac{3s}{2}.$$

Finally, this must be a solution as we know from the lectures that a solution must exist.

3. Let K be a continuous integral kernel. Let us consider the iterated kernels

$$K^{(1)}(s,t) = K(s,t), \quad K^{(n)}(s,t) = \int_t^s K(s,r)\, K^{(n-1)}(r,t)\, \mathrm{d}r,$$

which were defined in the lectures. Show that

$$K^{(n)}(s,t) = \int_t^s K^{(n-1)}(s,r)\, K(r,t)\, \mathrm{d}r.$$

Hint: Use induction on n.

Solution. We shall use induction on n as instructed. The claim holds trivially for $n = 2$.

Let us assume that the claim holds for some $n \geqslant 2$ so that

$$K^{(n)}(s,t) = \int_t^s K^{(n-1)}(s,r)\, K(r,t)\, \mathrm{d}r.$$

Then

$$\begin{aligned} K^{(n+1)}(s,t) &= \int_t^s K(s,r)\, K^{(n)}(r,t)\, \mathrm{d}r \\ &= \int_t^s K(s,r) \int_t^r K^{(n-1)}(r,u)\, K(u,t)\, \mathrm{d}u\, \mathrm{d}r \\ &= \int_t^s \int_u^s K(s,r)\, K^{(n-1)}(r,u)\, \mathrm{d}r\, K(u,t)\, \mathrm{d}u \\ &= \int_t^s K^{(n)}(s,u)\, K(u,t)\, \mathrm{d}u. \end{aligned}$$

4. Let us consider the *Fredholm integral equation of the second kind*

$$\varphi(s) - \lambda \int_a^b K(s,t)\, \varphi(t)\, \mathrm{d}t = f(s), \quad a \leqslant s \leqslant b, \tag{$*$}$$

where $K \in C\big([a,b]\times[a,b]\big)$, $f \in C\big([a,b]\big)$ and $\lambda \in \mathbb{C}$. Study what extra conditions are needed for the kernel K so that the ansatz

$$\varphi(x) = \sum_{n=0}^{\infty} \lambda^n \, \varphi_n(x)$$

used in the lectures would give a continuous solution to $(*)$. Will the solution then be unique?

Solution. We shall prove, using arguments similar to those used in the lectures for the Volterra equation of the second kind, that if

$$|\lambda| < \frac{1}{M}, \quad \text{where} \quad M = \max_{a\leqslant s\leqslant b} \int_a^b |K(s,t)| \, \mathrm{d}t,$$

then the Fredholm equation of the second kind has a unique solution $u \in C\big([a,b]\big)$, which is indeed given by the ansatz involving the iterations of integration against K: the ansatz was

$$\varphi(s) = \sum_{n=0}^{\infty} \lambda^n \, \varphi_n(s)$$

for all $s \in [a,b]$, where $\varphi_0 = f$ and

$$\varphi_n(s) = \int_a^b K(s,t) \, \varphi_{n-1}(s) \, \mathrm{d}t$$

for all $s \in [a,b]$ and $n \in \mathbb{Z}_+$.

The iterate $\varphi_n(s)$ can be estimated for $s \in [a,b]$ by

$$|\varphi_n(s)| \leqslant M^n \, m,$$

where

$$m = \max_{a\leqslant s\leqslant b} |f(s)| \, .$$

This is perhaps easiest to see through induction. By the definition of m, we certainly have $|\varphi_0(s)| \leqslant m$. Now, if $|\varphi_{n-1}(s)| \leqslant M^{n-1}\, m$ for all $s \in [a,b]$ for some $n \in \mathbb{Z}_+$, then

$$|\varphi_n(s)| = \left| \int_a^b K(s,t) \, \varphi_{n-1}(t) \, \mathrm{d}t \right| \leqslant \int_a^b |K(s,t)| \, \mathrm{d}t \, M^{n-1} \, m \leqslant M^n \, m.$$

Now we can prove that the infinite series converges uniformly for $s \in [a,b]$. This follows by estimating the tail of the series:

$$\left| \sum_{n>N} \lambda^n \, \varphi_n(s) \right| \leqslant \sum_{n>N} |\lambda|^n \, M^n \, m \leqslant m \sum_{n>N} |\lambda \, M|^n \xrightarrow[N\longrightarrow\infty]{} 0.$$

Thus φ is a well defined continuous function on $[a, b]$. Also, this allows us to integrate φ against K termwise giving

$$\lambda \int_a^b K(s,t)\,\varphi(t)\,\mathrm{d}t = \sum_{n=1}^{\infty} \lambda^n \varphi_n(s),$$

for all $s \in [a, b]$. Since the infinite series involved converge absolutely, a simple exchange of the order of summation shows that φ indeed solves the Fredholm equation of the second kind.

Finally, the solution is unique: If there was another solution $\psi \in C\big([a, b]\big)$, then the difference $\varphi - \psi$ would satisfy the homogeneous Fredholm equation of the second kind

$$\varphi(s) - \psi(s) = \lambda \int_a^b K(s,t)\,\big(\varphi(t) - \psi(t)\big)\,\mathrm{d}t$$

for $s \in [a, b]$. Let $|\varphi - \psi|$ obtain its maximum at a point $s_0 \in [a, b]$. Given the condition $|\lambda| < \frac{1}{M}$, we see that

$$|\varphi(s_0) - \psi(s_0)| \leqslant |\lambda|\, M\, |\varphi(s_0) - \psi(s_0)|\,.$$

Now $\varphi(s_0) - \psi(s_0) = 0$, for otherwise we would have

$$|\varphi(s_0) - \psi(s_0)| < |\varphi(s_0) - \psi(s_0)|\,.$$

But then $\varphi(s) = \psi(s)$ for all $s \in [a, b]$ and we have shown uniqueness.

5. Reduce the initial value problem

$$y^{(3)} + 2xy = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = 1$$

to an equivalent Volterra equation of the second kind.

Solution. The integral equation is obtained by repeatedly integrating the equation. The first integration gives simply

$$y''(x) - 1 + 2\int_0^x t\,y(t)\,\mathrm{d}t = 0.$$

The second integration gives first

$$y'(x) - x + 2\int_0^x \int_0^u t\,y(t)\,\mathrm{d}t\,\mathrm{d}u = 0,$$

and exchanging the integral signs gives

$$y'(x) - x + 2\int_0^x t \int_t^x \mathrm{d}u\,y(t)\,\mathrm{d}t = 0,$$

which simplifies to

$$y'(x) - x + 2\int_0^x t\,(x-t)\,y(t)\,\mathrm{d}t = 0.$$

In the same vein, integrating for the third time gives

$$y(x) - \frac{x^2}{2} + 2\int_0^x \int_0^u t\,(u-t)\,y(t)\,\mathrm{d}t\,\mathrm{d}u = 0.$$

Changing again the order of integration gives

$$y(x) - \frac{x^2}{2} + 2\int_0^x t \int_t^x (u-t)\,\mathrm{d}u\,y(t)\,\mathrm{d}t = 0,$$

which simplifies to

$$y(x) + \int_0^x t\,(x-t)^2\,y(t)\,\mathrm{d}t = \frac{x^2}{2},$$

which is a Volterra integral equation of the second kind.

Finally, we see that this Volterra equation implies the original initial value problem simply by repeatedly substituting $x = 0$ and differentiating, in the same way as in the solutions to the problems 1 and 2.

6. Solve the Volterra equation of the first kind

$$\int_1^s (s+t)\,\varphi(t)\,\mathrm{d}t = s^3 - 1.$$

Solution. Let us look again for a continuous solution of the integral equation. Certainly a solution must be defined in a neighbourhood of 1 in order for the integral equation to make sense. It will turn out that, in a small neighbourhood of 1, there is a unique solution, which will extend to all positive reals but tends to ∞ as $s \longrightarrow 0+$. Thus we will ultimately be looking for a continuous function $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$.

Differentiating the integral equation gives

$$2s\,\varphi(s) + \int_1^s \varphi(t)\,\mathrm{d}t = 3s^2.$$

This equation implies both the initial condition $\varphi(1) = \frac{3}{2}$ and the continuous differentiability of φ. Differentiating the equation again gives the differential equation

$$2\varphi(s) + 2s\,\varphi'(s) + \varphi(s) = 6s.$$

For $s \in \mathbb{R}_+$ this has the equivalent form

$$\frac{3}{2}\,s^{1/2}\,\varphi(s) + s^{3/2}\,\varphi'(s) = 3\,s^{3/2}.$$

This is just

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(s^{3/2}\,\varphi(s)\Big) = 3\,s^{3/2},$$

and so

$$s^{3/2}\,\varphi(s) = \frac{6}{5}\,s^{5/2} + C$$

for some constant C. Since $\varphi(1) = \frac{3}{2}$, we have $C = \frac{3}{10}$, and the only possible solution to the integral equation is

$$\varphi(s) = \frac{6s}{5} + \frac{3}{10}\,s^{-3/2}.$$

Finally, we can easily check that this really is a solution to the original integral equation:

$$\begin{aligned}
&\int_1^s (s+t)\left(\frac{6t}{5} + \frac{3}{10}\,t^{-3/2}\right)\mathrm{d}t \\
&= \int_1^s \left(\frac{6st}{5} + \frac{6t^2}{5} + \frac{3s}{10}\,t^{-3/2} + \frac{3}{10}\,t^{-1/2}\right)\mathrm{d}t \\
&= \left(\frac{3st^2}{5} + \frac{2t^3}{5} - \frac{3s}{5}\,t^{-1/2} + \frac{3}{5}\,t^{1/2}\right)\Bigg]_1^{t=s} \\
&= \frac{3s^3}{5} - \frac{3s}{5} + \frac{2s^3}{5} - \frac{2}{5} - \frac{3}{5}\,s^{1/2} + \frac{3s}{5} + \frac{3}{5}\,s^{1/2} - \frac{3}{5} = s^3 - 1.
\end{aligned}$$

7. Let us consider the Volterra equation of the first kind

$$\int_a^s K(s,t)\,\varphi(t)\,\mathrm{d}t = f(s), \tag{$*$}$$

where K and f are continuous. Let us assume that $K(s,s) = 0$ for all $s \in [a,b]$, and that the function K has continuous partial derivatives with respect to s up to order two. Formulate and prove a solvability result for the equation $(*)$.

Solution. Certainly f must be at least once continuously differentiable because the left-hand side is. Also, $f(a)$ must vanish. Since K vasnishes of the diagonal, differentiating the integral equation gives

$$\int_a^s \big(\partial_1 K\big)(s,t)\,\varphi(t)\,\mathrm{d}t = f'(s),$$

where ∂_1 denotes differentiation with respect to the first variable of $K(\cdot,\cdot)$.We now see that f' must be continuously differentiable and vanish at a, too. Differentiating the equation again gives

$$\big(\partial_1 K\big)(s,s)\,\varphi(s) + \int_a^s \big(\partial_1^2 K\big)(s,t)\,\varphi(t)\,\mathrm{d}t = f''(s).$$

Now, if $\big(\partial_1 K\big)(s,s) \neq 0$ for all s then this is a Volterra integral equation of the second kind:

$$\varphi(s) + \int\limits_a^s \frac{\big(\partial_1^2 K\big)(s,t)}{\big(\partial_1 K\big)(s,s)}\, \varphi(t)\,\mathrm{d}t = \frac{f''(s)}{\big(\partial_1 K\big)(s,s)}.$$

This equation always has a unique continuous solution by the results proved in the lectures. Also, multiplying this latter equation by $\big(\partial_1 K\big)(s,s)$ and then integrating the equation twice, we obtain the original Volterra equation of the first kind. We thus obtain a solvability result:

Proposition. *Let $K \in C\big([a,b]\times[a,b]\big)$ be twice continuously differentiable with respect to the first variable, and assume that $K(s,s) = 0$ and $\big(\partial_1 K\big)(s,s) \neq 0$ for all $s \in [a,b]$. Also, let $f \in C^2\big([a,b]\big)$ satisfy $f(a) = f'(a) = 0$. Then the Volterra integral equation of the first kind*

$$\int\limits_a^s K(s,t)\, \varphi(t)\,\mathrm{d}t = f(s)$$

has a unique solution $\varphi \in C\big([a,b]\big)$.