Let $y_n = x_n / n\|x_n\|_X$. Then $\|y_n\| = 1/n \to 0,\ n \to \infty$

but

$$\|Ay_n\|_Y = \frac{\|Ax_n\|_X}{n\|x_n\|_X} \geq 1$$

so A is not continuous at 0. ↯ □

We denote $\mathcal{L}(X,Y) = \{A\ ;\ A : X \to Y \text{ bnd. \& linear}\}$

The inf of all M satisfying (2.2.1) is denoted by $\|A\|$ and is the (operator) norm of A. This really defines a norm on $\mathcal{L}(X,Y)$ (Ex in HW3). Furthermore, if Y is complete (i.e. Y Banach) then also $\mathcal{L}(X,Y)$ is complete w.r.t. operator-norm: Assume $A_n \in \mathcal{L}(X,Y)$, $n \in \mathbb{N}$, is Cauchy i.e.

$$\forall \varepsilon,\ \exists n_\varepsilon \text{ s.t. } m,n > n_\varepsilon \Rightarrow \|A_m - A_n\| < \varepsilon .$$

Then $\forall x \in X$,

$$\|A_m x - A_n x\| \leq \|A_m - A_n\|\|x\| < \varepsilon\|x\| ,$$

so $(A_m x)$ is a Cauchy-sequence in $Y \Rightarrow \exists$ limit

$$Ax := \lim_{n\to\infty} A_n x .$$

It is easy to see that A is linear. It is also bounded: Choose n_0 s.t. $m,n > n_0 \Rightarrow \|A_m - A_n\| < 1$. Fix $m > n_0$, then $\forall n > n_0$,

$$\|A_n\| < 1 + \|A_m\|$$

and thus

$$\|A_n x\|_Y \leq (1 + \|A_m\|)\|x\|_X$$

But $\|Ax\|_Y = \lim \|A_n x\|_Y \leq (1+\|A_m\|)\|x\|_X$

$\therefore$ A bnd.

Now we only need to show that $\|A_n - A\| \to 0,\ n \to \infty$. Now let $m,n > n_\varepsilon$. Then

$$\|(A - A_n)x\|_Y = \lim_{m\to\infty} \|A_m x - A_n x\|_Y \leq \varepsilon\|x\|_X \quad \forall x \in X$$

$\Rightarrow \|A - A_n\| < \varepsilon$. Hence $\|A - A_n\| \underset{n\to\infty}{\to} 0$.

If $X = H_1$ and $Y = H_2$ are Hilbert spaces all this holds and esp. $\mathcal{L}(H_1, H_2)$ is a complete normed space i.e. a Banach space!

<u>Note</u>: $\mathcal{L}(H_1,H_2)$ is usually <u>not</u> a Hilbert-space even if H_1 and H_2 are.

In a Hilbert space there are two very special types of linear ops.

2.2.1. <u>Projections</u> (i.e. ortho-projections)

Recall the following in $\mathbb{R}^n$: let $M \subset \mathbb{R}^n$ be a subspace. Let $(m_1, \dots, m_k)$ be its orthonormal base i.e. $\|m_i\| = 1$ and m_i & m_j are orthogonal to each other if $i \neq j$. Then $\exists$ basis $(m_1, \dots, m_k, m_{k+1}, \dots, m_n)$ of whole $\mathbb{R}^n$ that is also orthonormal. Especially $\exists$ bnd map

$$P : \sum_{i=1}^{n} x_i m_i \longrightarrow \sum_{i=1}^{k} x_i m_i \in M$$

and if $M^\perp = \text{span}\{m_{k+1}, \dots, m_n\}$, a map

$$Q : \sum_{i=1}^{n} x_i m_i \longrightarrow \sum_{i=k+1}^{m} x_i m_i \in M^\perp$$

s.t. $Px \perp Qx\ \forall x$, $x = Px + Qx$;

P is the orthogonal proj. to M

Q is the " " to $M^\perp$.

This is still true in Hilbert-spaces (but usually not in Banach spaces) if we assume M is closed.
Note that in the finite dimensional case all subspaces are autom. closed. In Hilbert spaces this need not hold (let $M = C([0,1]) \subset L^2([0,1])$).

- First, if $x, y \in H$, H Hilbert, then by def.

$$\begin{cases} x \perp y & (x \,\&\, y \text{ orthogonal to each other}) \\ \text{iff} & \langle x,y\rangle = 0 \end{cases}$$

- If $M \subset X$ is a subspace, then we define the orthogonal complement $M^\perp$ as

$$M^\perp = \{ y \in H ;\ \langle x,y\rangle = 0 \ \forall x \in M \}$$
(even if M is not!)

- Note that $M^\perp$ is always a closed subspace:

if $\alpha, \beta \in \mathbb{C}$, $y_1, y_2 \in M^\perp$ then $\forall x \in M$

$$\langle x, \alpha y_1 + \beta y_2 \rangle = \bar\alpha \underbrace{\langle x, y_1\rangle}_{=0} + \bar\beta \underbrace{\langle x, y_2\rangle}_{=0} = 0$$

i.e. $\alpha y_1 + \beta y_2 \in M^\perp$.

Also, if $y_n \in M^\perp$, $y_n \to y \in H$, then $\forall x \in M$:

$$0 = \langle x, y_n\rangle \to \langle x,y\rangle \Rightarrow \langle x,y\rangle = 0 \text{ i.e. } y \in M^\perp.$$

<u>Thm.</u> If M is a closed subspace of H, then $\exists$ unique pair of linear maps P, Q s.t.

$$P : H \to M, \quad Q : H \to M^\perp$$

and

(P1) $\quad x = Px + Qx \qquad \forall x \in H$

Also

(P2) if $x \in M \Rightarrow Px = x,\ Qx = 0$
$\qquad x \in M^\perp \Rightarrow Px = 0,\ Qx = x$

(P3) $\|x - Px\| = \|Qx\| = \inf\{\|x-y\| ;\ y \in M\}$

(P4) $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$ (Pythagoras)

Pf: Omitted; see for example [Astala-Tylli: Funktionaalianalyysi], or [Rudin, Real & Complex, pp. 34-35 (at least in my edition :))]

2.2.2. Bounded linear functionals

Let's again start from $\mathbb{R}^n$; consider a linear map (a functional)

$$\lambda : \mathbb{R}^n \to \mathbb{R}.$$

How could one try to represent λ? Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$; then

$$\lambda(x) = \lambda\Big(\sum_1^n x_k e_k\Big) = \sum_1^n x_k \lambda(e_k) = \langle x, z_\lambda\rangle, \quad z_\lambda = (\lambda(e_1), \dots, \lambda(e_n)).$$

Hence λ is just inner product with z_λ, a fixed element of $\mathbb{R}^n$!

It is not wise to expect this to be true in the infinite dimensional case; for example if one equips $C([0,1])$ with sup-norm (hence it becomes a Banach space but not a Hilbert space), the space of linear functionals on $C([0,1])$ is the space of regular Borel measures on $[0,1]$ — a huge space, much larger than $C([0,1])$.

However on a Hilbert space things work out great:

<u>Thm.</u> (Riesz representation thm.) If $\lambda : H \to \mathbb{C}$ is a bounded linear functional i.e. λ linear and $\exists M > 0$ s.t.

$$|\lambda(x)| \le M\|x\| \quad \forall x \in H,$$

then $\exists$ <u>unique</u> $y \in H$ s.t.

$$\lambda(x) = \langle x, y \rangle \quad \forall x \in H.$$

Pf. This follows easily from $\exists$ projections. If $\lambda \equiv 0$, then $y = 0$ and this is unique. So assume $\lambda \not\equiv 0$. Let

$$M = \{x : \lambda(x) = 0\}.$$

M is a <u>closed</u> (since λ cont.) subspace of H. Also, $M^\perp \neq 0$ (Otherwise proj thm. $\Rightarrow M = H \Rightarrow \lambda = 0$?).

Choose $z \in M^\perp$, $\|z\| = 1$. Let

$$u := \lambda(x) z - \lambda(z) x.$$

Then

(i) $\quad \lambda(u) = \lambda(x)\lambda(z) - \lambda(z)\lambda(x) = 0 \Rightarrow u \in M.$

Hence $\langle u, z \rangle = 0$, and

$$\lambda(x) = \lambda(x)\underbrace{\langle z, z \rangle}_{=\|z\|^2=1} = \langle \lambda(x) z, z \rangle$$

$$\overset{(i)}{=} \langle \lambda(z) x, z \rangle = \lambda(z)\langle x, z \rangle$$

$$= \langle x, z\overline{\lambda(z)} \rangle.$$

So choosing $y = z\overline{\lambda(z)}$ we get existence. Uniqueness follows easily: If $\forall x \in H$:

$$\lambda(x) = \langle x, y_1 \rangle = \langle x, y_2 \rangle$$

$\Rightarrow \langle x, y_1 - y_2 \rangle = 0 \quad \forall x \in H$. Choose $x = y_1 - y_2$. Then

$$0 = \langle y_1 - y_2, y_1 - y_2 \rangle = \|y_1 - y_2\|^2 \Leftrightarrow y_1 = y_2. \ \square$$

2.3 <u>Neumann series</u> (Named after a 19^{th}-century German mathemat. Carl Neumann)

Let's get back to - or at least closer to - integral equations. Let $A \in \mathcal{L}(H,H)$, H Hilbert (actually the inner product plays no role for the main thm; it holds in any Banach space). We prove the following:

<u>Thm 2.3.1</u> Assume $\|A\| < 1$. Then the eqn given $y \in H$

$$x - Ax = y$$

has a unique solution $x \in H$ and

$$x = y + Ay + \dots + A^n y + \dots$$

infact, $I - A$ is invertible in $\mathcal{L}(H)$ and

$$I - A = I + A + \dots + A^n + \dots = \sum_{k=0}^{\infty} A^k$$

and this series converges in operator norm.

Pf. Let

$$B_{m,n} = \sum_{k=m}^{n} A^k \quad (m < n).$$

Now

$$\|B_{m,n}\| \le \sum_{k=m}^{n} \|A\|^k = \|A\|^m \sum_{k=0}^{n-m} \|A\|^k = \|A\|^m \frac{1 - \|A\|^{n-m+1}}{1 - \|A\|} \xrightarrow[m,n \to 0]{} 0$$

since $\|A\| < 1$. Hence the partial sums $\sum_{k=0}^{n} A^k$ form a Cauchy sequence in $\mathcal{L}(H,H)$, and $\exists$ limit

$\mathcal{L}(H,H) \ni B := I + A + \dots + A^n + \dots$

Let's prove that $(I-A)B = B(I-A) = I$. Let

$$B_n = I + A + \dots + A^n .$$

Then

$$(I-A)B_n = I - A^{n+1},$$

$$B_n(I-A) = I - A^{n+1},$$

so since $\|A\|<1$, $\|A^{n+1}\| \le \|A\|^{n+1} \xrightarrow[n\to\infty]{} 0$ and thus

$$I - A^{n+1} \xrightarrow[\mathcal{L}(H,H)]{} I$$

and

$$(I-A)B = \lim_n (I-A)B_n = I = \lim_n B_n(I-A) = B(I-A).$$

This proves the Thm. $\square$

["Small perturbations of identity are still invertible"]

Before moving to applications to integral eqns, let's prove the following important consequence of Thm. 2.3.1:

<u>Prop. 2.3.2.</u> Let H_1, H_2 be Hilbert. Let

$$\mathrm{Inv}(H_1,H_2) = \{A \in \mathcal{L}(H_1,H_2);\ \exists\ B \in \mathcal{L}(H_2,H_1) \text{ s.t. } AB = I_{H_2},\ BA = I_{H_1}\}.$$

The set of invertible elements

Then in the operator norm topology $\mathrm{Inv}(H_1,H_2)$ is an open subset of $\mathcal{L}(H_1,H_2)$.

Pf: Let $A \in \mathrm{Inv}(H_1,H_2)$, then for any $R \in \mathcal{L}(H_1,H_2)$

$$A+R = A(I_{H_1} + A^{-1}R),$$

and when $\|A^{-1}R\|<1$ is $I_{H_1} + A^{-1}R$ inv, and also $A+R$ as a product of inv. ops. But

$$\|A^{-1}R\| \le \|A^{-1}\|\,\|R\| < 1$$

when

$$\|R\| < (\|A^{-1}\|)^{-1}$$

Hence $\|R\| < (\|A^{-1}\|)^{-1} \Rightarrow A+R$ is inv. $\square$

Let's now apply this to integral ops: 1st we want to estimate the operator norm of an integral operator

$$Kf(x) = \int_\Omega K(x,y) f(y)\,dy, \quad x \in \Omega .$$

Here we assume that K is a meas. fund. on $\Omega\times\Omega$ s.t. functions

$$x \mapsto K(x,y), \quad y \mapsto K(x,y)$$

are resp cnt. a.a y or a.a. x.

<u>Prop.</u> 2.3.3 (Schur's lemma).

If

$$\sup_x \int_\Omega |K(x,y)|\,dy,\ \sup_y \int_\Omega |K(x,y)|\,dx \le M < \infty$$

then

$$K : L^2(\Omega) \to L^2(\Omega), \quad \|K\| \le M .$$

Pf. This is a straightforward application of Cauchy-Schwartz: Let $f \in C_0^\infty(\Omega)$ initially. Then

$$|Ku(x)| \le \int_\Omega |K(x,y)||u(y)|\,dy$$

$$\overset{C\text{-}S}{\le} \underbrace{\left(\int_\Omega |K(x,y)|^{1/2 \cdot 2}\,dy\right)^{1/2}}_{\le M^{1/2}} \left(\int_\Omega |K(x,y)|^{1/2 \cdot 2} |u(y)|^2\,dy\right)^{1/2}$$

Hence integrating over x,

$$\int_\Omega |Ku(x)|^2\,dx \le M \int_{\Omega\times\Omega} |K(x,y)||u(y)|^2\,dy\,dx$$

$$\le M \underbrace{\left(\sup_y \int_\Omega |K(x,y)|\,dx\right)}_{\le M} \int_\Omega |u(y)|^2\,dy$$

$$\le M^2 \|u\|^2_{L^2(\Omega)},$$

and thus

$$\|Ku\|_{L^2(\Omega)} \le M \|u\|^2_{L^2(\Omega)}. \square$$

Consider now the integral eqn:

$$(\ast) \qquad f(x) + \int_\Omega K(x,y) f(y)\,dy = g(x),$$

where K is a meas. function on $\Omega\times\Omega$.

Prop. 2.3.4. If

$$\sup_x \int_\Omega |K(x,y)|\,dy,\ \sup_y \int_\Omega |K(x,y)|\,dx < 1,$$

then the integral eqn $(\ast)$ has a unique solution $f \in L^2(\Omega)$ for all $g \in L^2(\Omega)$.

Pf. Apply Schur's lemma to

$$Kf(x) = \int_\Omega K(x,y) f(y)\,dy$$

and the result then follows from Neumann-series.

2.4. Compact operators in Hilbert spaces

The Neumann series was based on idea that "small perturbations of the identity are invertible" and we measured the size of an op using operator norm. There are other ways to control the size of an operator.

Recall that in $\mathbb{R}^n$ ($\&\ \mathbb{C}^n$) compact sets are precisely those which are closed and bounded. This is not true in the infinitely dimensional case: Let H be an inf. dimensional Hilbert space; then $\exists_{/\inf}$ seq. $(u_n)_{n=1}^\infty$ of lin. indep elements. Gram-Schmidt $\Rightarrow$ we may assume $(u_n)_1^\infty$ is orthonormal:

$$\|u_n\| = 1, \quad \langle u_n, u_m\rangle = 0,\ n \ne m.$$

Note that all

$$u_n \in B = \{x \in H;\ \|x\| \le 1\} \quad \text{which is closed \& bounded.}$$

On a cpt set of a metric space (normed spaces are metric) every sequence contains a convergent subsequence.

But for $(u_n)_1^\infty$:

$$n \ne m \Rightarrow \|u_n - u_m\|^2 = \langle u_n - u_m, u_n - u_m\rangle \overset{\langle u_n, u_m\rangle = 0}{=} \|u_n\|^2 + \|u_m\|^2 = 2$$

$$\Rightarrow \|u_n - u_m\| = \sqrt{2} \quad \therefore \quad (u_n)_1^\infty \text{ is not Cauchy} \Rightarrow \text{does not converge!}$$

So gen:

$$\{\text{cpt sets}\} \subsetneq \{\text{bnd. \& closed}\}.$$

<u>Def. 2.4.1</u> A bnd op $K : H_1 \to H_2$ (Hilbert) is <u>compact</u> if it maps every bnd set U in H_1 to a precompact set in H_2 i.e. if $\overline{K(U)}$ is compact $\forall\, U \subset H_1$ bounded.

$\mathcal{K}(H_1, H_2) = \{K \in \mathcal{L}(H_1, H_2);\ K \text{ cpt}\}$

<u>Rem.</u> i) All linear maps to finite dimensional spaces are compact.

ii) Let $B_{H_1} = \{x \in H_1 ;\ \|x\| = 1\}$ be <u>the</u> closed <u>unit ball</u> of H_1. Then it is enough to assume that $\overline{K(B_{H_1})}$ is compact.

We'll see in HW3 the following:

a) If $A, B : H_1 \to H_2$ are cpt then $A + B$ is cpt

b) If $K : H_1 \to H_2$ is cpt then

AK is cpt $\forall\ A \in \mathcal{L}(H_2, H)$

KB —"— $B \in \mathcal{L}(H, H_1)$

are cpt.

c) If $K_n \in \mathcal{K}(H_1, H_2)$ and $K \in \mathcal{L}(H_1, H_2)$, $\|K_n - K\| \xrightarrow[n\to\infty]{} 0$ then $K \in \mathcal{K}(H_1, H_2)$.

Now we'll prove the following:

<u>Prop. 2.4.2.</u> If $K \in \mathcal{L}(H_1, H_2)$ is cpt, then $K^* \in \mathcal{L}(H_2, H_1)$ is also cpt.

<u>Remark</u>

Recall that the adjoint A^* of $A \in \mathcal{L}(H_1, H_2)$ is def. by

$$\langle Ax, y\rangle_{H_2} = \langle x, A^* y\rangle_{H_1} \quad \forall\ x \in H_1, y \in H_2.$$

It is an easy consequence of Riesz-rep thm that A^* exists & is bounded:

Consider $\lambda_y : H_1 \to \mathbb{C}$, $x \mapsto \langle Ax, y\rangle$.

Then λ_y is a linear functional of H_1 $\Rightarrow$ $\exists\ z \in H_1$ s.t.

$$\lambda_y(x) = \langle x, z\rangle \quad , \quad \|z\|_{H_1} = \|\lambda_y\|$$
$$\langle Ax, y\rangle$$

Def. $A^* y = z$. Since

$$|\langle Ax, y\rangle| \le \|A\| \|x\| \|y\| \quad \forall\ x \in H_1$$

$\Rightarrow$ $\|\lambda_y\| \le \|A\| \|y\|$ and thus

$$\|A^* y\|_{H_2} \le \|A\| \|y\| \quad \text{and } A^* \text{ is bnd.}$$

Linearity follows from

$$\langle x, A^*(\alpha y_1 + \beta y_2)\rangle = \bar{\alpha}\langle Ax, y_1\rangle + \bar{\beta}\langle Ax, y_2\rangle$$
$$= \langle x, \alpha A^* y_1 + \beta A^* y_2\rangle.$$

So now let's prove that K^* is cpt if K is cpt.

Pf. We need the following fundamental result of functional analysis: If $(u_n)_{n=1}^{\infty}$ is a bnd sequence of a Hilbert space H, it contains a <u>weakly convergent</u>

subsequence $(u_{n_k})_{k=1}^{\infty}$ i.e. sequence

$$(\langle u_{n_k}, y\rangle)_{k=1}^{\infty}$$

is convergent in $\mathbb{C}$ $\forall\ y \in H$ [Pf: True even for Banach spaces & bnd linear funct. on them; see lectures on Funct. Analysis by Astala - Tylli]

Assume now that $(y_n)_{n=1}^{\infty}$ is a bnd sequence in H_2; We want to prove that $(K^* y_n)_1^{\infty}$ contains a convergent subsequence. Let

$$y_{n_k} \to y \quad \text{weakly in } H_2 .$$

Then

(i) $$\|K^* y_{n_k} - K^* y\|^2 = \langle K^* y_{n_k} - K^* y, K^* y_{n_k} - K^* y\rangle$$

$$= \langle KK^* y_{n_k} - KK^* y, y_{n_k} - y\rangle .$$

Now KK^* cpt, (y_n) bnd, so passing to a subsequence of (y_{n_k}) if necessary we may assume that

$$\|KK^*(y_{n_k} - y)\|_{H_2} \longrightarrow 0 ,$$

but since (y_{n_k}) bnd we have by (i)

$$\|K^* y_{n_k} - K^* y\|_{H_1}^2 \le C \|KK^*(y_{n_k} - y)\|_{H_2} \xrightarrow{n_k \to \infty} 0 \quad . \square$$

2.5 Integral ops with weakly singular kernels

Let's now study the compactness properties of

$$Ku(x) = \int_{\Omega} k(x,y)\, u(y)\, dy , \quad \Omega \Subset \mathbb{R}^n, \quad y, x \in \Omega .$$

We will use the following result:

Thm 2.5.1 Let $\mathcal{F}(H_1,H_2) = \{F \in \mathcal{L}(H_1,H_2) ; F$ has fin. dimensional image-space$\}$. Then (H_1, H_2 Hilbert-sp.)

$$\overline{\mathcal{F}}(H_1,H_2) = \mathcal{K}(H_1,H_2)$$

where the closure is w.r.t. operator-norm.

For the pf, see [Yosida; Functional Analysis].

Also

$$\overline{\mathcal{F}} \subset \mathcal{K}$$

is true also in ~~Hilbert~~ Banach-spaces, but the inclusion

$$\mathcal{K} \subset \overline{\mathcal{F}}$$

is not always. (Also $\overline{\mathcal{F}} \subset \mathcal{K}$ is easy to prove; see HW3)

Now we can prove:

Prop 2.5.2 If Ω bnd and $k \in C(\overline{\Omega} \times \overline{\Omega})$ then $K \in \mathcal{K}(L^2(\Omega), L^2(\Omega))$.

Pf. To keep notations simple, we consider only the case when $n=1$, $\Omega = (a,b) \subset \mathbb{R}$. Given $n \in \mathbb{N}$, let

Let

$$a_k = a + \frac{k(b-a)}{n}, \quad k = 0, \dots, n$$

$$\Omega_{k,l} = [a_k, a_{k+1}] \times [b_l, b_{l+1}], \quad k, l = 0, \dots, n-1$$

and choose $x_{k,l} \in \Omega_{k,l}$.

Then K unif. ~~bnd~~ cont. in $\bar\Omega \times \bar\Omega$ $\Rightarrow$

$$\sup_x \int_\Omega |(K - K^{(n)})(x,y)|\,dy, \quad \sup_y \int_\Omega |(K-K^{(n)})(x,y)|\,dx \to 0$$

where

$$K^{(n)}(x,y) = \sum_{k,l=0}^{n-1} K(x_{k,l})\, \chi_{\Omega_{k,l}}(x,y).$$

Hence, if

$$\mathcal{K}^{(n)} f(x) := \int_\Omega K^{(n)}(x,y) f(y)\,dy,$$

Schur $\Rightarrow$ $\mathcal{K}^{(n)} \to \mathcal{K}$ in $\mathcal{L}(L^2(\Omega), L^2(\Omega))$. But

$$\mathcal{K}^{(n)} f(x) = \sum_{k,l=0}^{n-1} K(x_{k,l}) \int_\Omega \chi_{\Omega_{k,l}}(x,y) f(y)\,dy$$

$$= \sum_{k,l=0}^{n-1} K(x_{k,l})\, \chi_{[a_k, a_{k+1}]}(x) \int_{a_l}^{a_{l+1}} f(y)\,dy$$

$\Rightarrow$

$$\mathcal{K}^{(n)} f \in \operatorname{span}\{\chi_{[a_0,a_1]}, \dots, \chi_{[a_{n-1},a_n]}\}$$

hence $\mathcal{K}^{(n)}$ cpt & $\mathcal{K}$ also.

The case of $\dim \Omega = n$ follows similarly approximating K by a linear combination of characteristic functions of cubes, we leave this to you ☺ □

Consider now a singular kernel of type

$$\text{(WS)} \qquad K(x,y) = \frac{Q(x,y)}{|x-y|^\alpha}, \qquad \alpha < n, \quad Q \in C(\bar\Omega \times \Omega)$$

and $\Omega \Subset \mathbb{R}^n$ bnd; a kernel of this type is called weakly singular. If $\alpha = n$, K is an example of so called Calderon–Zygmund kernel, if ~~we extend~~ we allow for $\Omega = \mathbb{R}^n$ and if $\alpha > n$, then the kernel is hypersingular.

We will consider only weakly singular kernels.

Prop. 2.5.3 If $\Omega \Subset \mathbb{R}^n$ bnd and K is weakly singular, then $\mathcal{K} : L^2(\Omega) \to L^2(\Omega)$ is compact.

Pf. Given $\varepsilon > 0$, let

$$K_\varepsilon(x,y) = \begin{cases} K(x,y), & |x-y| > \varepsilon, \\ \dfrac{Q(x,y)}{\varepsilon^\alpha}, & \text{if } |x-y| \le \varepsilon \end{cases}$$

Then K_ε is cont and hence

$$\mathcal{K}_\varepsilon f(x) = \int_\Omega K_\varepsilon(x,y) f(y)\,dy$$

defines a cpt op $L^2(\Omega) \to L^2(\Omega)$. Let

$$D_\varepsilon(x,y) := K(x,y) - K_\varepsilon(y) = \begin{cases} 0, & |x-y| > \varepsilon \\ Q(x,y)\left[\dfrac{1}{|x-y|^\alpha} - \dfrac{1}{\varepsilon^\alpha}\right], & |x-y| \le \varepsilon. \end{cases}$$

Now

$$\int_\Omega |K(x,y)|\,dy \le \left(\sup_{\Omega\times\Omega} Q\right) \int_\Omega |x-y|^{-\alpha}\,dy \le C_{\Omega,\alpha} \sup_{\bar\Omega \times \bar\Omega} Q$$

int. when $\alpha < n$.

$$\int_\Omega |K(x,y)|\,dx \le \left(\sup_{\Omega\times\Omega} Q\right) \int_\Omega |x-y|^{-\alpha}\,dx \le C_{\Omega,\alpha} \sup_{\bar\Omega\times\bar\Omega} Q$$

So $K : L^2(\Omega) \to L^2(\Omega)$ is bnd. Now $\forall x \in \Omega$:

$$\int_\Omega |D_\varepsilon(x,y)|\,dx \le \|Q\|_\infty \int_{\substack{|x-y|<\varepsilon \\ y\in\Omega}} \left|\frac{1}{|x-y|^\alpha} - \frac{1}{\varepsilon^\alpha}\right| dy$$

$$\le \|Q\|_\infty \left(\int_0^\varepsilon r^{n-1-\alpha}\,dr + C\varepsilon^{n-\alpha} \right) \qquad \mathrm{vol}\{|x-y|\le\varepsilon\} \sim \varepsilon^n$$

$$= \|Q\|_\infty \left(\Big/_0^\varepsilon r^{n-\alpha} + C\varepsilon^{n-\alpha} \right) = O(\varepsilon^{n-\alpha}) \xrightarrow[\varepsilon\to 0]{} 0$$

Similarly

$\forall y$: $\displaystyle\int_\Omega |D_\varepsilon(x,y)|\,dy \le C\varepsilon^{n-\alpha}$

Hence $D_\varepsilon = K - K_\varepsilon$ $L^2(\Omega) \to L^2(\Omega)$ with norm $\sim C\varepsilon^{n-\alpha}$ and

hence K is cpt as a norm limit of cpt ops. □

III FREDHOLM-YHTÄLÖT JA KOMPAKTIEN OPERAATTORIEN SPEKTRI ← UPS: again in Finnish.

3.1. Definitions

Let $A \in \mathcal{L}(H_1, H_2)$; H_1, H_2 Hilbert. We define

$\ker A = \{x \in H_1 ; Ax = 0\}$ "kernel or null-space of A"

$\operatorname{im} A = \{y \in H_2 ; \exists x \in H_1 \text{ s.t. } Ax = y\}$

"image or range of A"

Note: A cont $\Rightarrow$ $\ker A$ is always a closed subspace.

$\operatorname{im} A$ might not be: Let $A : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$,

$$A : (a_n)_1^\infty \longmapsto (e^{-n} a_n)_1^\infty .$$

Then $\|A\| = 1/e$ and assume $\tilde{H} := \operatorname{im} A$ is closed.

If $x = (x_n)_1^\infty \in \tilde{H}^\perp$, then

$$0 = \langle (x_n), (e^{-n} a_n) \rangle = \sum x_n \bar{a}_n e^{-n} \quad \forall\, (a_n) \in \ell^2(\mathbb{C})$$

Hence (choose $a_n = \delta_{k,n}$) $x_k e^{-k} = 0 \;\forall k \Leftrightarrow x_k = 0$

$\therefore \tilde{H}^\perp = \{0\} \Leftrightarrow \tilde{H} = \ell^2(\mathbb{C})$. This cannot

↑ we assume $\tilde{H}$ closed

be true: let $b_n = 1/n$, then $(b_n) \in \ell^2(\mathbb{C})$,

and $A(a_n)_1^\infty = (b_n)_1^\infty \Leftrightarrow a_n = e^n b_n = e^n/n$.

But $(a_n)_1^\infty \notin \ell^2(\mathbb{C})$.

So

FACTS OF LIFE I. [image of a linear op is not generally a closed subspace]

Under some assumptions this will hold. To this end we define

Def 3.1.1 i) Let $K \in \mathcal{K}(H_1, H_2)$. Then an eqn

$$Kx = y, \quad y \in H_2,$$

is a Fredholm equation of 1^{st} kind

ii) if $H = H_1 = H_2$ and $K \in \mathcal{K}(H_1, H_2)$, an equation

$$(\mathbb{I}_H - K)x = y, \quad y \in H_2,$$

is a Fredholm equation of 2^{nd} kind.

Like with Volterra-eqns, Fredholm eqns of 2^{nd} kind are easier to analyze than 1^{st} kind. This we will now do.

3.2. Fredholms alternative

In this section, $K : H \to H$ is a fixed cpt, H Hilbert.

Lemma 3.2.1 $\ker(\mathbb{I} - K)$ is finite dimensional.

Pf. Assume $\dim \ker(\mathbb{I} - K) = +\infty$. Then $\exists$ infinite ON-set $u_k \in \ker(\mathbb{I} - K)$, $k = 1, 2, \dots$ i.e.

↑ use Gram-Schmidt

$$\|u_k\| = 1, \quad \langle u_k, u_\ell \rangle = 0 \quad \text{when } k \neq \ell.$$

Now (we've seen this before) if $k \neq \ell$,

$$\|u_k - u_\ell\|^2 = \langle u_k - u_\ell, u_k - u_\ell \rangle = \underbrace{\|u_k\|^2}_{=1} + \underbrace{\|u_\ell\|^2}_{=1} - \overbrace{\langle u_k, u_\ell \rangle}^{=0} - \overbrace{\langle u_\ell, u_k \rangle}^{=0}$$

$= 1$

So $(u_k)_1^\infty$ cpt bnd seq; Also

$$(\mathbb{I} - K) u_k = 0 \iff K u_k = u_k. \text{ Hence}$$

$$\|K u_k - K u_\ell\|^2 = \|u_k - u_\ell\|^2$$

and hence $\nexists$ a convergent subsequence of $(K u_n)_1^\infty$. ↯. $\square$

<u>Lemma 3.2.2.</u> $\mathrm{im}(\mathbb{I} - K)$ is closed.

Pf. Let's first prove that $\exists\ \gamma > 0$ (const) s.t.

$(\perp)$ $\quad \forall u \in \ker(\mathbb{I} - K)^\perp : \ \|u - Ku\| \geq \gamma \|u\|.$

Assume $(\perp)$ does not hold for any $\gamma > 0$. Then $\forall k\ \exists$ $u_k \in H$, $\|u_k\| = 1$,

$(1/k)$ $\qquad \|u_k - K u_k\| < 1/k.$

Hence $u_k - K u_k \xrightarrow{H} 0$. Since (u_k) bnd, $\exists$ weakly conv. subsequence (u_{k_j}) ; let $u = w\text{-}\lim u_{k_j}$.

But then (after possibly moving to ~~a~~ a subseq. once more)

$$K \text{ cpt} \implies K u_{k_j} \xrightarrow{H} K u \overset{(1/k)}{\implies} u_{k_j} \longrightarrow Ku$$

so $(1/k)$ again gives $Ku = u$. Hence $u \in \ker(\mathbb{I} - K)$ and thus

$$0 = \langle u_{k_j}, u \rangle = 0, \quad j = 1, 2, \cdots,$$

But $u_{k_j} \xrightarrow{w} u \implies$

$$0 = \lim_j \langle u_{k_j}, u \rangle = \|u\|^2,$$

but since $\|u_{k_j}\| \to 1$, $u_{k_j} \xrightarrow{H} u \implies \|u\| = 1$ ↯.

Hence $(\perp)$ holds.

Assume $\|v_k - v\| \to 0$, $v_k = u_k - K u_k$; and we can take $u_k \in \ker(\mathbb{I} - K)^\perp$. Then

$$\|v_k - v_\ell\| = \|\underbrace{(u_k - u_\ell)}_{\in \ker(\mathbb{I} - K)^\perp} - K(u_k - u_\ell)\| \geq \gamma \|u_k - u_\ell\|$$

$\implies (u_k)$ is a Cauchy seq. in $H \implies \exists\ u = \lim u_k$. But Then

$$(\mathbb{I} - K) u = \lim (u_k - K u_k) = \lim v_k = v$$

$\implies v \in \mathrm{im}(\mathbb{I} - K)$. $\square$

Before the next result we need the following general fact:

<u>Prop. 3.2.3.</u> If $A \in \mathcal{L}(H_1, H_2)$, H_1, H_2 Hilbert then

$$\overline{\mathrm{im}(A)} = \ker(A^*)^\perp.$$

Pf. It is enough to show that

$$\langle Ax, y\rangle = 0 \quad \forall x \in H_1,\ y \in \ker(A^*).$$

But $A^* y = 0 \Rightarrow$

$$0 = \langle x, A^* y\rangle = \langle Ax, y\rangle \quad \forall x \in H_1. \ \square$$

Prop. 3.2.4. If $K \in \mathcal{K}(H, H)$, then

$$\operatorname{im}(\mathbb{I}-K) = \ker(\mathbb{I}-K^*)^\perp$$

Pf. Since $\operatorname{im}(\mathbb{I}-K)$ is closed, this follows from the prev. prop. $\square$

This is often said as "$\operatorname{im}(\mathbb{I}-K)$ has finite codimension". Namely, since $\operatorname{im}(\mathbb{I}-K)$ is closed we can write

$$H = \underbrace{\ker(\mathbb{I}-K^*)}_{\text{finite dimensional}} \overset{\perp}{\oplus} \ker(\mathbb{I}-K^*)^\perp = \ker(\mathbb{I}-K^*) \overset{\perp}{\oplus} \operatorname{im}(\mathbb{I}-K)$$

Prop. 3.2.5 If $K \in \mathcal{K}(H,H)$, then

$$\ker(\mathbb{I}-K) = \{0\} \iff \operatorname{im}(\mathbb{I}-K) = H.$$

i.e. $\boxed{\mathbb{I}-K \text{ is inj} \iff \mathbb{I}-K \text{ is surj.}}$

Pf. Assume $\ker(\mathbb{I}-K) = \{0\}$, but

$$\operatorname{im}(\mathbb{I}-K) =: H_1 \subsetneq H$$

is a proper closed subspace of H. Let now $H_2 := (\mathbb{I}-K)H_1$. This is again a closed subspace of H_1 & H. Assume $H_2 = H_1$. Then $\forall x \in H\ \exists\ y \in H_1$ s.t.

$$(\mathbb{I}-K)^2 x = x_1 = (\mathbb{I}-K)y_1 \quad \text{for a unique } y_1 \in H$$

($(\mathbb{I}-K)$ inj)

$\mathbb{I}-K$ inj $\Rightarrow$ $(\mathbb{I}-K)x = y_1$. Hence $H_1 = H$ $\lightning$.

Thus

$$H \supsetneq H_1 \supsetneq H_2.$$

Generally letting $H_k = (\mathbb{I}-K)H_{k-1}$, $k = 2, 3, \cdots$, we have a strictly decreasing sequence of closed subspaces

$$H \supsetneq H_1 \supsetneq H_2 \supsetneq \cdots \supsetneq H_k \supsetneq H_{k+1} \supseteq \cdots$$

Choose $u_k \in H_k$, $\|u_k\| = 1$ and $u_k \perp H_{k+1}$. ← We can do this since $\forall k$ $H_{k+1} \subsetneq H_k \Rightarrow H_k \neq \{0\}$.

Then

$$(*)\ Ku_k - Ku_\ell = -\underbrace{(u_k - Ku_k)}_{\in H_{k+1} \subset H_{\ell+1}} + \underbrace{(u_\ell - Ku_\ell)}_{\in H_{\ell+1}} + (u_k - u_\ell).$$

Assume $k > \ell$. Then $u_k \in H_k \subset H_{\ell+1}$. Hence

$$-(u_k - Ku_k) + (u_\ell - Ku_\ell) + u_k \subset H_{\ell+1}$$

$$u_\ell \in H_{\ell+1}^\perp,\ \|u_\ell\| = 1$$

So $(*) \Rightarrow$

$\|Ku_k - Ku_\ell\|^2 \geq \|u_\ell\|^2 = 1$ $\therefore$ This is a contradiction since K cpt. Thus

(im) $\quad \operatorname{im}(\mathbb{I}-K) = H.$

Conversely assume (im) holds. Prop. 3.2.4 $\Rightarrow$

$$\ker(\mathbb{I}-K^*) = \{0\}$$

So $\quad \operatorname{im}(\mathbb{I}-K^*) = H$

$\overset{\shortparallel}{\ker(\mathbb{I}-K)^\perp} \Rightarrow \ker(\mathbb{I}-K) = \{0\}.\ \square$

<u>Prop. 3.2.6</u> $\dim\ker(\mathbb{I}-K) = \dim\ker(\mathbb{I}-K^*)$.

Pf. It is enough to prove

(i) $\dim\ker(\mathbb{I}-K) \geq \dim \operatorname{im}(\mathbb{I}-K)^\perp$,

for $\operatorname{im}(\mathbb{I}-K)^\perp = \ker(\mathbb{I}-K^*)$,

so (i) implies

(ii) $\dim\ker(\mathbb{I}-K) \geq \dim\ker(\mathbb{I}-K^*)$

and the claim follows from this by applying (ii) to K^*.

To prove (i) we argue by contradiction: assume

(iii) $\dim\ker(\mathbb{I}-K) < \dim\operatorname{im}(\mathbb{I}-K)^\perp$.

Let

$$A : \ker(\mathbb{I}-K) \to \operatorname{im}(\mathbb{I}-K)^\perp$$

be bnd, linear, injective and not onto; here we use (iii). Extend

$A : H \to \operatorname{im}(\mathbb{I}-K)^\perp$ s.t. $A\big|_{\ker(\mathbb{I}-K)^\perp} = 0$.

A is finite dimensional $\Rightarrow$ $K+A$ is cpt.

Also

$$0 = (\mathbb{I}-(K+A))u \iff u = Ku + Au \iff \underbrace{u-Ku}_{\in \operatorname{im}(\mathbb{I}-K)} = \underbrace{Au}_{\in \operatorname{Im}(\mathbb{I}-K)^\perp}$$

$\Rightarrow 0 = u - Ku = Au \Rightarrow \begin{cases} u \in \ker(\mathbb{I}-K) \\ Au = 0 \end{cases} \xRightarrow[\text{on } \ker(\mathbb{I}-K)]{A \text{ inj.}} u = 0$.

Hence $\mathbb{I}-(K+A)$ is inj. Then Prop. 3.2.5 $\Rightarrow$

$$\operatorname{im}(\mathbb{I}-(K+A)) = H.$$

This is not true: choose $v \in \operatorname{im}(\mathbb{I}-K)^\perp$, $v \notin \operatorname{im}(A)$.

Then if

$$\underbrace{u-Ku}_{\in \operatorname{im}(\mathbb{I}-K)} - Au = v \in \operatorname{im}(\mathbb{I}-K)^\perp \xRightarrow[\text{thm}]{\text{Proj.}} -Au = v \notin \operatorname{im}(A) \text{ ↯}. \quad \square$$

<u>3.2.7 Fredholm's alternative</u> We can summarize the above as follows: Let $K \in \mathcal{K}(H,H)$;

(A) if $\mathbb{I}-K$ is inj., then it is onto, and conversely if $\mathbb{I}-K$ is onto, then $\mathbb{I}-K$ is inj.

(B) if $\mathbb{I}-K$ is not inj. let $k < \infty$ be $k = \dim\ker(\mathbb{I}-K)$;

Then $k = \dim\ker(\mathbb{I}-K^*) = \dim\operatorname{im}(\mathbb{I}-K)^\perp$ and hence eqn

$$(\mathbb{I}-K)u = v$$

has a solution u if and only if $v \perp \operatorname{im}(\mathbb{I}-K)^\perp$ i.e. $\exists$ ON vectors $\beta_1, \dots, \beta_k \in \operatorname{im}(\mathbb{I}-K)^\perp$ s.t.

$$\langle v, \beta_\ell \rangle = 0, \quad \ell = 1, \dots, k.$$

Also u is unique only modulo a subspace of dimension k.

3.3. <u>Spectrum of cpt operators</u>

Recall that if A is a $n\times n$ matrix, $\lambda \in \mathbb{C}$ is an <u>eigenvalue</u> if $A - \lambda\mathbb{I}$ is <u>not</u> invertible. For a general $n\times n$-matrix, eigenvalues are the (complex) roots of the polynomial (of degree n)

$$0 = \det(A - \lambda\mathbb{I}),$$

hence there are precisely n eigenvalues (counting multiplicities)

For a general $A \in \mathcal{L}(H,H)$, H Hilbert, we define:

<u>3.3.1 Def.</u> (i) The <u>resolvent set</u> $\rho(A)$ of A is

$$\rho(A) = \{\lambda \in \mathbb{C};\ A - \lambda \mathbb{I} \text{ is a bijection } H \to H\}$$

(ii) The spectrum of A, $\sigma(A)$ is

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

<u>Rem.</u> Note that Open Mapping Thm $\Rightarrow (A - \lambda \mathbb{I})^{-1}$ is bnd if $\lambda \in \rho(A)$.
Hence we could equally well define

$$\rho(A) = \{\lambda \in \mathbb{C};\ A - \lambda \mathbb{I} \text{ is invertible}\}.$$

<u>Ex. 3.3.2.</u> Let $H = L^2([-a,a])$, $a > 0$ and let

$$A : f \mapsto x f;$$

then $\|A\| \leq a$; let $\lambda \in \mathbb{C}$;

$$(A - \lambda) f = (x - \lambda) f = 0 \Leftrightarrow f = 0 \text{ a.e.}$$

i.e. $A - \lambda$ is always inj. But given $g \in L^2([-a,a])$,

$$(A - \lambda) f = g \Leftrightarrow f(x) = \frac{g}{x - \lambda} \in L^2 \ \forall g \text{ iff } |\lambda| > a.$$

Hence

$$\rho(A) = \{\lambda \in \mathbb{C};\ |\lambda| > a\}$$

$$\sigma(A) = \{\lambda \in \mathbb{C};\ |\lambda| \leq a\}.$$

Note that the situation is very different from the finite dimensional case: For matrices $\lambda - A$ inj $\Leftrightarrow \lambda - A$ surj; hence $\lambda - A$ <u>not</u> invertible $\Leftrightarrow \exists\, x \neq 0$ (eigenvector) s.t.

$$Ax = \lambda x.$$

This is <u>not</u> the case for the operator of Ex. 3.3.2: injectivity is <u>always</u> valid but the surjectivity fails!

To be able to be more precise, let:

<u>Def. 3.3.3.</u> $\lambda \in \sigma(A)$ is an <u>eigenvalue</u> of A if $\exists$ $x \in H$, $x \neq 0$ s.t.

$$(\ddot\smile) \qquad Ax - \lambda x \neq 0.$$

We call $x \neq 0$ satisfying $(\ddot\smile)$ an <u>eigenvector</u> (corresponding to eigenvalue λ). Let

$$\sigma_p(A) = \{\lambda \in \mathbb{C};\ \lambda \text{ eigenvalue}\}.$$

Set $\sigma_p(A)$ is <u>the point spectrum of A</u>.

So for the operator in example 3.3.2,

$$\sigma(A) = [-a, a], \quad \sigma_p(A) = \emptyset.$$

For compact A one might think that $\sigma(A)$ behaves more like the spectrum of a matrix (since they are norm limits of ops with finite range). This is true to a certain extent:

<u>Thm. 3.3.4.</u> Assume H is an infinite dimensional Hilbert-space. Then if $K \in K(H,H)$,

(i) $0 \in \sigma(K)$

(ii) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$

(iii) $\sigma(K)\setminus\{0\}$ is either finite or a sequence converging to 0.

Pf. (i) Assume $0 \notin \sigma(K)$. Then $\exists\, K^{-1} \in \mathcal{L}(H,H)$ and thus $\mathbb{I} = KK^{-1}$ is cpt. Since $\dim H = \infty$ this is a contradiction.

(ii) Let $0 \neq \lambda \in \sigma(K)$. If $\ker(\lambda\mathbb{I} - K) = \{0\}$, Fredholm alt. $\Rightarrow \operatorname{im}(\lambda\mathbb{I} - K) = H \Rightarrow \lambda \notin \sigma(K)$ ↯.

(iii) Assume now $(\lambda_k)_{k=1}^{\infty}$ is a sequence of <u>distinct</u> elements of $\sigma(K)\setminus\{0\}$. We will prove the following:

(lim) $\quad \lambda = \lim_{k\to\infty} \lambda_k \Rightarrow \lambda = 0.$

This will imply that if $\sigma(K)\setminus\{0\}$ is infinite, it consists of a sequence tending to 0: first, if $\lambda > \|K\|$, then

$$\lambda I - K = \lambda\left(I - \frac{1}{\lambda}K\right)$$

is invertible by the Neumann-series. Since $\sigma(K) \subset [-\|K\|, \|K\|]$, & set of eigenvalues outside $\{0\}$ is inf; it has accumulation pts. By (lim) 0 is the only possible accumulation pt, proving the claim. So let's prove (lim): Given $\lambda_k \in \sigma(K)\setminus\{0\} = \sigma_p(K)\setminus\{0\}$, $\exists\, w_k \neq 0$ s.t.

$$Kw_k = \lambda_k w_k.$$

Let

$$H_k = \operatorname{span}\{w_1, \dots, w_k\}.$$

Now we need a

<u>Lemma 3.3.5.</u> If $(\lambda_j)_1^{\infty}$ are distinct, then $(w_k)_1^{\infty}$ are linearly independent.

We will prove this in a moment; let's 1st prove the Thm. using this. Lemma 3.3.5 $\Rightarrow$

$$H_k \subsetneq H_{k+1}.$$

Also

$$(K - \lambda_k\mathbb{I})H_k \subseteq H_{k-1}$$

since

$$(K - \lambda_k\mathbb{I})w_i = \begin{cases} 0, & i = k \\ Kw_i - \lambda_k w_i \subset H_{k-1}, & i < k \end{cases}$$

Choose now

$$u_k \in H_k,\ u_k \in H_{k-1}^{\perp},\ \|u_k\| = 1.$$

If $k > \ell$,

$$H_{\ell-1} \subsetneq H_\ell \subseteq H_{k-1} \subsetneq H_k,$$

and

$$\left\|\frac{1}{\lambda_k}Ku_k - \frac{1}{\lambda_\ell}Ku_\ell\right\| = \Bigg\|\underbrace{\frac{Ku_k - \lambda_k u_k}{\lambda_k}}_{\in H_{k-1}} - \underbrace{\frac{(Ku_\ell - \lambda_\ell u_\ell)}{\lambda_\ell}}_{\in H_{\ell-1} \subseteq H_{k-1}} + \underbrace{u_k}_{\in H_\ell^{\perp}} - \underbrace{u_\ell}_{\in H_\ell}\Bigg\|$$

$$\geq \|u_k\| = 1$$

If now $\lambda_k \longrightarrow \lambda \neq 0$ this contradicts the compactness of K. □

Let's prove Lemma 3.3.5:

<u>Pf of Lemma 3.3.5</u>: Assume $\sum_{i=1}^{n} a_i w_i = 0$, $a_i \in \mathbb{C}$.

where $K\omega_k = \lambda_k \omega_k$, $\omega_k \neq 0$ & λ_k distinct.

Then $\forall \ell$

$$0 = K^\ell \left(\sum_{i=1}^{n} a_i \omega_i\right) = \sum_{i=1}^{n} \lambda_k^\ell a_i \omega_i$$

and hence taking $\ell = 0, \dots, n-1$ we get

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}}_{=\Lambda^{n-1}} \begin{bmatrix} a_1 w_1 \\ \vdots \\ a_n w_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

But

$$\det \Lambda^{n-1} = \prod_{i<j} (\lambda_i - \lambda_j) \quad \text{(Van der Monde)}$$
$$\neq 0$$

so $a_1 w_1 = \dots = a_n w_n = 0 \Rightarrow a_k = 0 \ \forall k$. $\square$

We can prove little more if we assume that K is self-adjoint.

<u>Prop. 3.3.6</u> If $K = K^*$ then all eigenvalues of K are real, and ~~eigenfunc~~ eigenvectors corresponding to disjoint eigenvalues are orthogonal.

Pf. Exc. like in the linear algebra course:

Assume $Kx = \lambda x$, $x \neq 0$. Then $K = K^* \Rightarrow$

$$\lambda \langle x, x\rangle = \langle Kx, x\rangle = \langle x, Kx\rangle = \bar\lambda \langle x, x\rangle$$

$\overset{x\neq 0}{\Rightarrow} \lambda = \bar\lambda$ i.e. λ real.

Also, if $Kx = \lambda x$, $Ky = \mu y$, $x, y \neq 0$, $\mu \neq \lambda$, then $K = K^* \Rightarrow$

$$\lambda \langle x, y\rangle = \langle Kx, y\rangle = \langle x, Ky\rangle = \bar\mu \langle x, y\rangle \overset{\mu\in\mathbb{R}}{=} \mu \langle x, y\rangle$$

$\Leftrightarrow (\lambda - \mu)\langle x, y\rangle = 0$. But $\lambda \neq \mu \Rightarrow \langle x, y\rangle = 0$. $\square$

<u>3.4. Dirichlet eigenvalues</u>

As an application of all the above, we study the following important question.

Let $\Omega \Subset \mathbb{R}^2$ be a bounded domain. We want to understand what kind of "eigenvibrations" Ω can have. So, let's start with the wave eqn in Ω:

$$(\square) \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } \Omega;$$

here u is the "small" vertical displacement of Ω from equilibrium state $u \equiv 0$. (We scale units s.t. wave-speed $\equiv 1$ in Ω.

By an "eigenmode" or "eigenvibration" we mean the following:

1. Ω is clamped at $\partial\Omega$ i.e. $u|_{\partial\Omega} = 0 \ \forall t$.
2. u has freq. $\omega > 0$ i.e.

$$u(t, x) = e^{i\omega t} U(x).$$

Now plugging (2) to (□) gives

$$-\omega^2 e^{i\omega t} U(x) + e^{i\omega t} \Delta U = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{Laplace-op}$$

$$\Leftrightarrow \left((-\Delta) - \omega^2\right) U = 0 .$$

Hence (in some vague sense) U must be an eigenfunction of $-\Delta$ with eigenvalue ω^2.

Also (i) $\Rightarrow$ $U|_{\partial\Omega} = 0$.

Hence we want to know what we can mathematically say about those $\omega > 0$ that have $U \neq 0$ satisfying

$$\begin{cases} \Delta U = \omega^2 U \\ U|_{\partial\Omega} = 0 . \end{cases}$$

However, let's start with the following:

2. Weak solutions for the ~~Dirichlet~~ Poisson problem

We want to solve

$$(P) \qquad \begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 . \end{cases}$$

This is a classical, well understood problem that however has closed form solutions only in (generally) symmetric special domains Ω. We want to use Hilbert-space methods, and the 1st grand question is then how to formulate (P) in some Hilbert space?