Integral equations

HW 5, fall 2013

1. Define

$$K(s,t) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2}\{\cos(k+1)s \sin kt - \sin(k+1)s \cos kt\}, \ \ s,\ t \in \mathbb{R}.$$

   Prove that the integral operator with kernel $K(s,t)$ on $L^2([0,2\pi])$ has no eigenvalues.

2. Let $\Phi_k(x) = e^{ik|x|}/4\pi|x|$ and assume that $\varphi \in C_0(\mathbb{R}^3)$. Prove that

$$u(x) = \int \Phi_k(x-y)\,\varphi(y)\,dy$$

   satisfies Sommerfeld's radiation condition.

3. Assume that $\Omega \subset \mathbb{R}^3$ is a bounded C^2–domain with a connected complement. Assume that $u \in C^2(\mathbb{R}^3 \setminus \Omega)$ satisfies Sommerfeld's radiation condition and solves $\Delta u + k^2 u = 0$. Show that if $x \in \mathbb{R}^3 \setminus \Omega$, then

$$u(x) = \int_{\partial\Omega} u(y)\frac{\Phi_k(x-y)}{\partial\nu(y)} - \frac{\partial u(y)}{\partial\nu(y)}\Phi_k(x-y)\,dS(y).$$

   Here ν is the exterior unit normal to Ω.

4. With otherwise the same assumptions as in Exercise 2, show that if $x \in \Omega$, then

$$0 = \int_{\partial\Omega} u(y)\frac{\Phi_k(x-y)}{\partial\nu(y)} - \frac{\partial u(y)}{\partial\nu(y)}\Phi_k(x-y)\,dS(y).$$

5. Consider the scattered field as defined in the lectures:

$$u_s(x) = -k^2 \int_{\mathbb{R}^3} \Phi_k(x-y)\,m(y)u(y)\,dy,$$

   where $u \in C_0^2(\mathbb{R}^3)$. Show that u has the asymptotics

$$u_s(x) = \frac{e^{ik|x|}}{|x|}u_\infty(x/|x|) + \mathcal{O}(|x|^{-2}), \ \ |x| \to \infty.$$

   The function u_∞ is called the *far field pattern* of u_s. Give an explicit formula for it.

6. Let now $u_i(x) = e^{ik\langle x,d\rangle}$, where $|d| = 1$. Let $u_\infty(x/|x|, d)$ be the far field pattern of the corresponding scattered field. Show that it satisfies the following *reciprocity relation*

$$u_\infty(x/|x|, d) = u_\infty(-d, -x/|x|).$$

Explain this result in terms of incoming and outgoing directions.