Integral equations

HW 4, fall 2013

1. Define
$$x_+^a = \begin{cases} x^a, \ x > 0 \\ 0, \ x \le 0. \end{cases}$$
Determine those values $a \in \mathbb{R}$ for which x_+^a has a weak derivative in the sense that we defined in the lectures.

   For the next three exercises we assume that H is a **real** Hilbert space. Especially the inner product $\langle \cdot, \cdot \rangle$ is an $\mathbb{R}$–bilinear map on $H \times H$.

2. Assume that $B : H \times H \to \mathbb{R}$ is a real bilinear map for which there exists constants M, $m > 0$ such that
$$|B(u, v))| \le M\|u\| \, \|v\|, \quad u, \, v \in H,$$
and
$$m\|u\|^2 \le B(u, u), \quad u \in H.$$
Prove that there is a unique bounded linear operator $A : H \to H$ such that
$$B(u, v) = \langle Au, v\rangle, \quad u, \, v \in H.$$

3. Prove that the operator A constructed above is a bijection.

4. Prove now the *Lax-Milgram Theorem*: If B is as above and $\lambda : H \to \mathbb{R}$ is a bounded linear functional, then there exists a unique element $u \in H$ such that for all $v \in H$ we have
$$B(u, v) = \lambda(v).$$

Let now $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider the linear partial differential operator

$$L = -\Delta + \sum_{k=1}^{n} b^k(x) \frac{\partial}{\partial x_k} + c(x)$$

where the real valued functions b_k and c are continuous in $\overline{\Omega}$.

5. Define the bilinear form

$$B(u,v) = \int_\Omega \langle \nabla u, \nabla v \rangle + \sum_1^n b^k \frac{\partial u}{\partial x_k} v + cuv \, dx$$

on $H_0^1(\Omega) \times H_0^1(\Omega)$. Prove that B satisfies the so-called energy estimates: there exists positive constants M, m and C such that

$$|B(u,v)| \leq M \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$$

and

$$m\|u\|^2_{H_0^1(\Omega)} \leq B(u,u) + C\|u\|^2_{L^2(\Omega)}$$

for all u, $v \in H_0^1(\Omega)$.

6. Apply the previous exercise to study the weak solvability on $H_0^1(\Omega)$ of the boundary value problem

$$Lu + \mu u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

for a large enough constant μ.

7. Show that the set of Dirichlet eigenvalues of Δ on $\Omega \subset \mathbb{R}^d$ is invariant under rotations, reflections and translations of Ω.

8. Given $\lambda > 0$ and $\Omega \subset \mathbb{R}^d$, let $\lambda\Omega = \{\lambda x; \, x \in \Omega\}$. What can you say about the Dirichlet–eigenvalues of $\lambda\Omega$?

For the next two exercises fix a bounded domain $\Omega \subset \mathbb{R}^d$, let

$$C^2_\partial(\Omega) = \{u \in C^2(\Omega) \cap C(\overline{\Omega}); \, u|_{\partial\Omega} = 0\}$$

and define

$$\lambda_1 = \inf_{w \in C^2_\partial(\Omega)} \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}}.$$

9. Assume $u \in C^2_\partial(\Omega)$ is such that

$$\lambda_1 = \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}},$$

i.e we attain the minimum at u. Prove that λ_1 is a Dirichlet eigenvalue of $-\Delta$ on Ω with eigenvalue u. **Hint**: Given any $v \in C^2_\partial(\Omega)$ study the function

$$f(\varepsilon) = \frac{\|\nabla(u + \varepsilon v)\|^2_{L^2(\Omega)}}{\|u + \varepsilon v\|^2_{L^2(\Omega)}},$$

at origin.

10. Prove that the $\lambda_1 \leq \lambda$ for all Dirichlet eigenvalues λ of $-\Delta$ on Ω.