Integral equations

HW 3, fall 2013

1. Assume $K : H_1 \to H_2$ is a compact operator between Hilbert spaces. Given bounded linear maps $A : H \to H_1$ and $B : H_2 \to H$, where H is again Hilbert, prove that KA and BK are compact. Also, prove that the sum $K_1 + K_2$ of two compact operators K_1, $K_2 : H_1 \to H_2$ is compact.

2. Assume $K_n : H_1 \to H_2$, $n = 1, 2, \dots$, are compact and that $\|A - K_n\| \to 0$ as $n \to \infty$. Here $A \in \mathcal{L}(H_1, H_2)$. Prove that A is compact.

3. Assume (a_n) is a sequence of complex numbers converging to zero. Consider the linear map
$$A : \ell^2 \to \ell^2, \quad (x_n) \mapsto (a_n x_n).$$
Prove that K is compact. **Hint:** use the previous exercise with suitable operators K_n having finite dimensional image spaces.

4. Give an example of a bounded linear operator between Hilbert spaces whose image is not a closed subspace.

5. Assume $K \in \mathcal{L}(H_1, H_2)$ and that for some positive integer n_0 we know that K^{n_0} is compact. What can you say about $\ker(I - K)$?

6. Assume A, $B \in \mathcal{L}(H, H)$ commute, i.e. $AB = BA$. If AB is invertible, what can you say about the invertibility of A and B?

7. Consider the integral operator
$$\mathcal{K}u(x) = \int_a^b K(x, y)u(y)\, dy, \quad x \in (a, b).$$
Assume that $K \in L^2([a, b])$. Prove that $\mathcal{K}$ is compact $L^2([a, b]) \to L^2([a, b])$.

8. Prove that a compact operator $K : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ is a norm limit of finite dimensional operators. **Hint**: Let Q_n be the orthogonal projection to $\operatorname{span}(e_1, \dots, e_n)$, where (e_i) is the standard orthonormal basis of $\ell^2(\mathbb{C})$. Let $K_n = Q_n K$ and prove that $\|K - K_n\| \to 0$ by considering a suitable finite covering of the compact set $\overline{K(B)}$ where B is the closed unit ball of $\ell^2(\mathbb{C})$.

9. Let's define the shift operator $S : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by

$$(Sx)_n = \begin{cases} 0, \ n = 0 \\ x_{n-1}, \ n = 1, 2, \ldots \end{cases}$$

Here $x = (x_n)_{n=1}^{\infty}$. Also let $M : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ be defined by

$$(Mx)_n = (n+1)^{-1} x_n.$$

Show that the product $T = MS$ is a compact operator that has no eigenvalues. Hence the spectrum consists only of $\{0\}$.