15.11.2012

Exercise 8

1. Let X_t^i denote the r.v. of ith Bernoulli trial in generation t.

It takes values 0 and 1, which can be thought as the # of A_1 chosen to be the parent of the ith offspring (i.e. the parent is either of type A_1 or not $\Rightarrow$ parent is A_2). If the parent is A_1, the offspring is also A_1. Let x be the "realization" of X, s.t $x_1 = 0$ and $x_2 = 1$.

Recall, $$Var[X_t^i] = \sum_{j=1}^{2} (x_j - E[X_t^i])^2 Pr\{X_t^i = x_j\}$$

As $E[X_t^i] = p$, where p is the frequency of A_1 in the parent generation (see Lecture notes), then

$$Var[X_t^i] = (0-p)^2(1-p) + (1-p)^2 p$$
$$= p^2(1-p) + p - 2p^2 + p^3$$
$$= p - p^2 = p(1-p)$$

2. We have (see also the previous exercise)

$$Var[K_1] = \sum_j (x_j - E[K_1])^2 Pr\{K_1 = x_j\}$$

From the lecture notes we get

$$E[K_1] = Np$$

and from the fact that

$$\Pi_{ij} = \begin{cases} p(1-p) & \text{if } j = i+1 \\ p(1-p) & \text{if } j = i-1 \\ p^2 + (1-p)^2 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

we have

$$Var[K_1] = ((pN-1) - pN)^2 p(1-p) +$$
$$(pN - pN)^2 (p^2 + (1-p)^2) +$$
$$((pN+1) - pN)^2 p(1-p)$$
$$= 2p(1-p)$$

3. See Lecture notes

We have $H_1 = 2\mathbb{P}_1(1-\mathbb{P}_1)$ and $H_0 = 2p_0(1-p_0)$

(a) $$\begin{aligned} E[H_1] &= E[2\mathbb{P}_1(1-\mathbb{P}_1)] \\ &= 2\,E[\mathbb{P}_1 - \mathbb{P}_1^2] \\ &= 2(E[\mathbb{P}_1] - E[\mathbb{P}_1^2]) \\ &= 2(E[\mathbb{P}_1] - E[\mathbb{P}_1]^2 - Var[\mathbb{P}_1]) \\ &= 2\left(\frac{E[K_1]}{N} - \left(\frac{E[K_1]}{N}\right)^2 - Var\left[\frac{K_1}{N}\right]\right) \\ &= 2\left(p_0 - p_0^2 - \frac{1}{N^2} Var[K_1]\right) \\ &= 2\left(p_0(1-p_0) - \frac{1}{N} p_0(1-p_0)\right) \\ &= 2p_0(1-p_0)\left(1 - \frac{1}{N}\right) \\ &= H_0\left(1 - \frac{1}{N}\right) \end{aligned}$$

(b) $$\begin{aligned} E[H_1] &= 2(E[\mathbb{P}_1] - E[\mathbb{P}_1]^2 - Var[\mathbb{P}_1]) \\ &= 2\left(p_0(1-p_0) - \frac{1}{N^2} 2p_0(1-p_0)\right) \\ &= 2p_0(1-p_0)\left(1 - \frac{2}{N^2}\right) \end{aligned}$$

4. From lecture notes

$$\Pr\{K_{t+1}=j\} = \sum_{i=0}^{N} \Pr\{K_t=i\}\pi_{ij} \quad . \qquad (1)$$

Using the fact that $\sum_{j=0}^{N} j\pi_{ij} = i$ (check!), we have
see also below

$$E[K_{t+1}] \overset{\text{def.}}{=} \sum_{j=0}^{N} j \Pr\{K_{t+1}=j\} \overset{(1)}{=} \sum_{j=0}^{N} j \sum_{i=0}^{N} \Pr\{K_t=i\}\pi_{ij}$$

$$= \sum_{i=0}^{N} \Pr\{K_t=i\} \sum_{j=0}^{N} j\pi_{ij}$$

$$= \sum_{i=0}^{N} i \Pr\{K_t=i\} = E[K_t]$$

As this is true for all t we have

$$E[K_{t+1}] = E[K_t] = \dots = E[K_0]$$

Alternatively, we can write directly

$$E[K_{t+1}] = \sum_{i=0}^{N} \underbrace{E[K_{t+1} | K_t = i]}_{i} \Pr\{K_t=i\} = E[K_t]$$