29.11.2012

Exercises 9.

1. T is a continuous random variable. We have

(a)
$$E[T] = \int_0^\infty t f(t)\,dt$$

where

$$f(t) = \lambda e^{-\lambda t}.$$

Setting $g(t) = t$, $h'(t) = f(t)$, we integrate by parts

$$\int_0^\infty g(t)\,h'(t)\,dt = \Big/_0^\infty g(t)\,h(t) - \int_0^\infty g'(t)\,h(t)\,dt$$

$$\Leftrightarrow$$

$$\int_0^\infty t f(t)\,dt = \Big/_0^\infty -te^{-\lambda t} + \int_0^\infty e^{-\lambda t}\,dt$$

$$= 0 - \Big/_0^\infty \frac{1}{\lambda}e^{-\lambda t} = 0 - \left[0 - \frac{1}{\lambda}\right] = \frac{1}{\lambda}$$

(b) T is a disrete random variable.

$$E[T] = \sum_{t=1}^\infty t(1-p)^{t-1}p = p\sum_{t=1}^\infty t(1-p)^{t-1}$$

* gives how many times we sum $(1-p)^{t-1}$

$$= p\left[\sum_{t=1}^\infty (1-p)^{t-1} + \sum_{t=2}^\infty (1-p)^{t-1} + \ldots\ldots\right]$$

$$\sum_{k=0}^\infty ar^k = \frac{a}{1-r}$$

$$= p\left[\frac{1}{p} + \underbrace{\left(\frac{1}{p} - 1\right)}_{\frac{1-p}{p}} + \underbrace{\left(\frac{1}{p} - 1 - (1-p)\right)}_{\frac{(1-p)^2}{p}} + \ldots\right]$$

$$= 1 + (1-p) + (1-p)^2 + \ldots$$

$$= \sum_{t=0}^\infty (1-p)^t = \frac{1}{p}$$

(2) $G_{i,j} = \dfrac{S_i^{[j]} N_{[j]}}{N^i}$, $1 \le j \le i$ (see Lecture notes)

Suppose $j < i-1$, then

$$G_{i,j} = \underbrace{\frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-j+1}{N}}_{j \text{ elements}} \cdot \underbrace{\frac{1}{N} \cdots \frac{1}{N}}_{i-j \text{ elements}} \cdot S_i^{(j)}$$

since $j < i-1 \Rightarrow$ at least two elements!

$$= \left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right) \cdots \left(1-\frac{j-1}{N}\right) \cdot \underbrace{\frac{1}{N} \cdots \frac{1}{N}}_{(*)} \cdot S_i^{(j)}$$

$$= O\left(\frac{1}{N^2}\right)$$

since, as mentioned above (*) has at least two elements, i.e. it is at least of order $\frac{1}{N^2}$.

(3) $G_{i,i-1}$ is the probability that two out of i lineages coalesce in one time-step.

In the Moran model this happens, when the individual chosen for reproduction and death are not the same AND that the reproducing individual (parent P) and its offspring (O) are among the i lineages.

We need to calculate

$$(*)\quad Pr\{O \text{ in sample} \cap P \text{ in sample}\} =$$

$$= 1 - \left[Pr\{O \text{ not in sample}\} + Pr\{P \text{ not in sample}\} - Pr\{O \text{ not in sample} \cap P \text{ not in sample}\}\right]$$

- $Pr\{O \text{ not in sample}\} = \frac{N-1}{N} \cdot \frac{N-2}{N-1} \cdots \frac{N-1-(i-1)}{N-(i-1)}$

$$= \frac{N-i}{N} = Pr\{P \text{ not in sample}\}$$

- $Pr\{O \text{ not in sample} \cap P \text{ not in sample}\} =$

$$= \frac{N-2}{N} \cdot \frac{N-3}{N-1} \cdots \frac{N-2-(i-1)}{N-(i-1)} = \frac{(N-i)(N-1-i)}{N(N-1)}$$

$$\Rightarrow (*)\quad Pr\{O \text{ in sample} \cap P \text{ in sample}\} =$$

$$= 1 - 2\frac{N-i}{N} + \frac{(N-i)(N-1-i)}{N(N-1)} =$$

$$= \frac{N(N-1) - (N-i)(N-1+i)}{N(N-1)} = \frac{Ni - i - Ni + i^2}{N(N-1)}$$

$$= \frac{i(i-1)}{N(N-1)}$$

As the probability that an individual chosen for reproduction is different than chosen for death is

$$1-\frac{1}{N}$$

we have

$$G_{i,i-1} = \left(1-\frac{1}{N}\right)\frac{i(i-1)}{N(N-1)} = \binom{i}{2}\frac{2}{N^2}.$$