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Exercise 5

1 (3 points)

Reciprocity: $c^*_\alpha \tilde{m}_{\alpha\beta} = c^*_\beta \tilde{m}_{\beta\alpha}$ (1)

(2) $$c^0_\beta = \sum_\alpha c^*_\alpha \tilde{m}_{\alpha\beta} \overset{(1)}{=} \sum_\alpha c^*_\beta \tilde{m}_{\beta\alpha} = c^*_\beta \underbrace{\sum_\alpha \tilde{m}_{\beta\alpha}}_{=1} = c^*_\beta$$

From $c^*_\alpha \tilde{m}_{\alpha\beta} = c^0_\beta m_{\beta\alpha}$ and (1) & (2) we get

$c^*_\alpha \tilde{m}_{\alpha\beta} = c^0_\beta m_{\beta\alpha} \overset{(2)}{=} c^*_\beta m_{\beta\alpha}$, on the other hand from (1)

$c^*_\alpha \tilde{m}_{\alpha\beta} = c^*_\beta \tilde{m}_{\beta\alpha}$

$\Rightarrow \quad c^*_\beta \tilde{m}_{\beta\alpha} = c^*_\beta m_{\beta\alpha}$

$\Leftrightarrow \quad \underline{\underline{\tilde{m}_{\alpha\beta} = m_{\alpha\beta}}}$ (3)

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a) From $c^*_\alpha \tilde{m}_{\alpha\beta} = c^0_\beta m_{\beta\alpha}$ follows, that

(4) $$\tilde{m}_{\beta\alpha} = \frac{c^0_\alpha}{c^*_\beta} m_{\alpha\beta} \overset{\text{Deakin}}{=} \begin{cases} \frac{c^0_\alpha}{c^*_\beta} \mu c^*_\beta = \mu c^0_\alpha & , \alpha \neq \beta \\ \frac{c^0_\alpha}{c^*_\beta}(1-\mu+\mu c^*_\alpha) \overset{\alpha=\beta}{=} \frac{c^0_\beta}{c^*_\beta}(1-\mu) + \mu c^0_\beta & \end{cases}$$

Since $$1 = \sum_\alpha \tilde{m}_{\beta\alpha} = \sum_{\alpha\neq\beta} \mu c^0_\alpha + \frac{c^0_\beta}{c^*_\beta}(1-\mu) + \mu c^0_\beta =$$

$$= \mu \sum_\alpha c^0_\alpha + \frac{c^0_\beta}{c^*_\beta}(1-\mu) = \mu + \frac{c^0_\beta}{c^*_\beta}(1-\mu) \quad \Rightarrow$$

$$1-\mu = \frac{c^o_\beta}{c^*_\beta}(1-\mu) \overset{(5)}{\Leftrightarrow} c^*_\beta = c^o_\beta \quad \text{for } \mu \in [0,1[$$

when $\mu = 1$ we get the Levene model as a special case (exercise 3)

b) $$c^*_\beta \tilde{m}_{\beta\alpha} \overset{(4)\&(5)}{=} c^*_\beta \mu c^*_\alpha = c^*_\alpha \tilde{m}_{\alpha\beta}$$, i.e. reciprocity

(3) Levene model assumes that migration is <u>independent</u> of the deme of origin, i.e. there exists constants μ_α with $\sum_\alpha \mu_\alpha = 1$ s.t.

(6) $$\tilde{m}_{\alpha\beta} = \mu_\beta \quad \text{for every } \alpha, \beta \in G$$

Then, we have $$m_{\beta\alpha} = \frac{c^*_\alpha \tilde{m}_{\alpha\beta}}{\sum_\gamma c^*_\gamma \tilde{m}_{\gamma\beta}} \overset{(6)}{=} \frac{\mu_\beta c^*_\alpha}{\mu_\beta \sum_\gamma c^*_\gamma} = c^*_\alpha$$

for every $\alpha, \beta \in G$.

This is a special case of Deakin model ($\mu = 1$)

$\Rightarrow c^o_\beta = c^*_\beta$

By $p'_{i,\alpha} = \sum_\beta m_{\alpha\beta} p^*_{i,\beta} = \sum_\beta c^*_\beta p^*_{i,\beta}$,

$p'_{i,\alpha}$ is independent of α, after one round of migration allele frequencies are the same in all demes

$\Rightarrow$ sufficient to study $p = (p_1, \dots, p_k)^T$

Thus

$$p_i' = p_i \sum_\alpha c_\alpha^* \frac{w_{i,\alpha}}{\overline{w}_\alpha} \qquad i \in K$$

Assuming soft selection ($c_\alpha = c_\alpha^*$) we get

$$p_i' = p_i \sum_\alpha c_\alpha \frac{w_{i,\alpha}}{\overline{w}_\alpha}$$

Consider two alleles A_1 and A_2 and write the frequency of A_1 with p.

Suppose two demes of equal size, i.e. $c_1 = \frac{1}{2} = c_2$.

To obtain condition for protected coexistence (PC) it is enough to consider the protection of one of the alleles, as the model is symmetric for A_1 and A_2. It is hence enough to calculate the stability of $p=0$.

Write $p' = F(p) = p\,f(p)$.

$p=0$ is unstable when

$$\left.\frac{\partial p'}{\partial p}\right|_{p=0} = f(0) > 1 \iff$$

$$\frac{1}{2}\frac{w_{1,1}(0)}{\overline{w}_1(0)} + \frac{1}{2}\frac{w_{1,2}(0)}{\overline{w}_2(0)} = \frac{1}{2}\frac{1-hs}{1-s} + \frac{1}{2}\frac{1-hs}{1} > 1$$

$$\iff \boxed{h < \frac{1}{2-s}} \quad \text{for} \quad 0 < s < 1$$

This is the condition for PC