Mathematical theory of population genetics

Exercises 3.

1. The selection equation

$$p_i' = p_i \frac{V_i}{\bar{V}}, \qquad \text{for } i = 1, \ldots, k, \tag{1}$$

where $V_i = \sum_j V_{ij} p_j$ and $\bar{V} = \sum_{i,j} V_{ij} p_i p_j$ are as in the lecture notes, can also be derived by explicitly writing out the mating process (using for example a mating table, see lecture notes) and by assuming equal mating probability and expected number of offspring for all the mating pairs. Mating is assumed to happen after the phase of selection and hence the frequency of $A_i A_j$ during the mating season is P_{ij}^* (see lecture notes).

Let us denote with $P_{kl}^* Q_{ij,kl}$ the probability that a female (male) $A_i A_j$ mates with a male (female) of type $A_k A_l$ (for example, $Q_{ij,kl}$ may denote the probability of mating given that individual of type $A_i A_j$ has encountered $A_k A_l$; however, in general, the interpretation of Q can be more complex, and it is only the multiplication $P^* Q$ which gives a probability). Further, let us denote with $F_{ij,kl}$ the expected number of offspring of a couple $A_i A_j$, $A_k A_l$ and denote with $R_{ij,kl \to mn}$ the (Mendelian) probability that parents $A_i A_j$ and $A_k A_l$ produce offspring of type $A_m A_n$. Then, for example, the expected number of offspring produced by a female $A_i A_j$ with a male $A_k A_l$ is simply $P_{kl}^* Q_{ij,kl} F_{ij,kl}$.

Also, note that $Q_{ij,kl} = Q_{ij,lk}$ (doesn't matter from which parent the allele is inherited, applies also to $F_{ij,kl}$), but in general $Q_{ij,kl} \neq Q_{kl,ij}$.

(a) (2 points) What is the expected number of offspring of type $A_m A_n$ produced by a female $A_i A_j$ with a male $A_k A_l$?

(b) (2 points) What is the total expected number of offspring produced by a female $A_i A_j$?

(c) (2 points) What is the total expected number of offspring of type $A_m A_n$ produced by a female $A_i A_j$?

(d) (4 points) What is the frequency of genotype $A_m A_n$ in the next generation? (Remember to normalize so that frequencies add up to 1 !)

Now, consider two alleles A_1 and A_2.

(e) (2 points) Write out the genotype frequencies of the next generation, $P'_{11}, 2P'_{12}, P'_{22}$.

(f) (4 points) Suppose that all the encounters are equally likely to end up in mating and that all matings result in equal number of offspring, that is, suppose that $Q = Q_{ij,kl}$ and $F = F_{ij,kl}$ for all i, j, k, l. Show that the selection equation (1) is recovered.