Differential Equation I
Excercise 3, fall 2012

1. A water tank is of a size 16m $\times$ 10m $\times$ 3m. Initially it is full of 3 per cent brine.
(a) Totally fresh water enters the tank at the rate $1.5\,m^3/min$, and the well stirred brine leaves the tank at the same rate. When does the concentration of salt in the tank descend to the level of 2 per cent?
(b) Otherwise the same situation but brine that enters the tank, is of 1.5 per cent.

2. A fish population of a lake was estimated to be 10000 individuals in 1990 and 5000 individuals in 2000. Let us model the fish population $p(t)$ by the logistic equation $\dot{p}(t) = rp(t)\big(1-p(t)/K\big)$. Suppose that the parameter r has an estimated value 0.1 (the unit of time is a year), but the tolerance K of environment is unknown.
(a) Determine K, (b) predict the population in 2010.
Remark. If also r is unknown, at least three known values of population are needed. Consideration yields then a system of equations (an usual system, not differential equations) that (mostly) has to be solve by some numerical method.

3. Solve the IVP

$$\dot{x} + 4tx = 2t\sqrt{x}, \quad x(0) = 1.$$

4. Solve the IVP

$$y' = ay - by^4, \quad y(0) = c,$$

where $a, b, c > 0$. Determine also the limit $\lim_{x\to\infty} y(x)$ of the solution.

5. Consider the SIR-model of infectious diseases, the pair (2.16),

$$\frac{ds}{dt}(t) = -\alpha R_0 s(t)i(t), \quad \frac{di}{dt}(t) = \alpha R_0 s(t)i(t) - \alpha i(t).$$

Suppose $0 < s(0), i(0) < 1$. Show that $i(t), s(t) > 0$ for all $t \geq 0$.
Tips. You can suppose the solution exists on $[0, \infty[$. For any solution functions $i(t)$ and $s(t)$ you can fix them alternately and apply the existence and uniqueness Theorem 1.2 separately to the equations (2.16).

6. Extend the problem 5: show that

$$(a) \quad \exists s_\infty = \lim_{t\to\infty} s(t) > 0, \quad (b) \quad \exists i_\infty = \lim_{t\to\infty} i(t) = 0.$$

Tips. Use Problem 5 and Equation (2.18), finally Lemma 2.1.