1 Introduction

We review some motivating examples. References are chapter I of [2] and chapter II of [1]. For background on dynamical systems and elementary application of multiscale perturbation theory to them see chapter VII of [3].

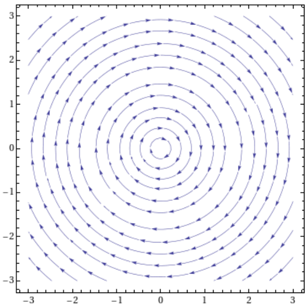
2 An exactly integrable model

The harmonic oscillator is the paradigmatic example of integrable models in classical physics. It is described by the linear second order ODE

$$\frac{d^2x}{dt^2} + \omega^2\, x = 0 \tag{2.1}$$

The motion is periodic along energy preserving trajectories

$$\mathcal{E} = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{\omega^2\, x^2}{2} \tag{2.2}$$



The figure is the stream plot of the harmonic oscillator for $\omega = 1$. If for example we consider the initial data

$$x_t|_{t=0} = x_o \qquad \& \qquad \left.\frac{dx_t}{dt}\right|_{t=0} = 0 \tag{2.3}$$

we have almost readily

$$x_t = x_o \cos(\omega\, t) \tag{2.4}$$

Let us now perturb the frequency ω by a factor $\sqrt{1+\varepsilon}$ with $\varepsilon \ll 1$. We have, if the initial conditions (2.3) are still specified by

$$x_t^{(\varepsilon)} = x_o \cos(\omega\, t\, \sqrt{1+\varepsilon}) \approx x_o \cos(\omega\, t\left(1 + \frac{\varepsilon}{2}\right)) \tag{2.5}$$

Furthermore if $\omega\, t \ll 1/\varepsilon$ we can also write

$$x_t^{(\varepsilon)} = x_o \left\{\cos(\omega\, t) - \frac{\varepsilon\, t}{2}\sin(\omega\, t) + O(\varepsilon^2)\right\} \tag{2.6}$$

We, thus observe that a perturbative expansion in ε may give rise to non-periodic approximate solutions! To see the genesis of this phenomenon in more details let us derive (2.6) from perturbation theory.

2.1 Perturbative expansion

We wish now to solve

$$\frac{d^2x}{dt^2} + \omega^2\, x = -\varepsilon\, \omega^2\, x \tag{2.7}$$

by seeking the solution in the form of the power series

$$x_t = \sum_{n=0}^{\infty} \varepsilon^n \, x_t^{(n)} \tag{2.8}$$

The zeroth and first order are then

$$\ddot{x}_t^{(0)} + \omega^2 \, x_t^{(0)} = 0 \tag{2.9a}$$

$$\ddot{x}_t^{(1)} + \omega^2 \, x_t^{(1)} = -\omega^2 \, x_t^{(0)} \tag{2.9b}$$

$$\vdots \tag{2.9c}$$

Since the initial data do not depend upon ε we have

$$x_t^{(0)}|_{t=0} = x_o \qquad \& \qquad \dot{x}_t^{(0)}|_{t=0} = 0 \tag{2.10}$$

and

$$x_t^{(0)}|_{t=0} = 0 \qquad \& \qquad \dot{x}_t^{(0)}|_{t=0} = 0 \qquad \forall \, n \, > \, 0 \tag{2.11}$$

We thus have that $x_t^{(0)}$ is specified by (2.4) while

$$\ddot{x}_t^{(1)} + \omega^2 \, x_t^{(1)} = -x_o \, \cos\left(\omega \, t\right) \tag{2.12}$$

Taking into account the boundary conditions the solution up to time T is specified by the *Cauchy Green function* of the harmonic oscillator (see appendix (A.1))

$$x_t^{(1)} = -\omega \int_0^T dt' G_C(t,t') \, x_s^{(0)} \tag{2.13}$$

where

$$G_C(t,t') = \frac{\sin[\omega(t-t')]}{\omega} \, H_{[0]}(t-t') \tag{2.14}$$

where $H_{[a]}$ stands for the Heaviside step-function

$$H_{[a]}(t) = \begin{cases} 1 & t \, > \, 0 \\ a & t \, = \, 0 \\ 0 & t \, < \, 0 \end{cases} \tag{2.15}$$

The integral gives

$$x_t^{(1)} = -\omega \, x_o \Im \int_0^t dt' \, e^{\imath \, \omega \, (t-t')} \frac{e^{\imath \, \omega \, t'} + e^{-\imath \, \omega \, t'}}{2} = -x_o \Im \frac{e^{\imath \, t} \, t \, \omega + \sin(\omega \, t)}{2 \, \omega} \tag{2.16}$$

whence we recover

$$x_t^{(1)} = -\frac{x_o \, \omega \, t}{2} \sin(\omega \, t) \tag{2.17}$$

We will refer to polynomially growing terms contributing to perturbative approximations of periodic orbits as *secular terms*. The occurrence of secular terms entails that perturbation theory holds only for times small enough. In such a case we try to interpret secular terms as the result of the expansion of some "slowly varying" periodic function. In the present case we know that for $\varepsilon\omega \, t \, \ll \, 1$

$$\sin\left(\frac{\varepsilon \, \omega \, t}{2}\right) \approx \frac{\varepsilon \, \omega \, t}{2} \tag{2.18}$$

Can we justify within perturbation theory resummations such as (2.18)? In order to do so we need to understand the mechanism responsible for the origin of terms polynomial in t in perturbation theory.

3 Inversion of operators and the origin of saecular terms

3.1 Linear algebra

The prototype problem is to solve for $\boldsymbol{x}$ the linear algebraic equation

$$\mathsf{A}\,\boldsymbol{x} = \boldsymbol{f} \tag{3.1}$$

where $\boldsymbol{x}, \boldsymbol{f} \in \mathbb{R}^d$ and A is an $\mathbb{R}^d \times \mathbb{R}^d$ matrix. In components (3.1) reads

$$\sum_{j=1}^{d} \mathsf{A}_{i\,j} x_j = f_i \tag{3.2}$$

If A is non singular

$$\det \mathsf{A} \neq 0 \tag{3.3}$$

the solution of the problem is unique and is specified by

$$\boldsymbol{x} = \mathsf{A}^{-1}\,\boldsymbol{f} \tag{3.4}$$

The problem (3.2) remains, however, well-posed also in the case of singular matrices, if some further hypotheses are added. Let us suppose

$$\mathrm{dimKer}\mathsf{A} = 1 \tag{3.5}$$

and

$$\mathsf{A}\,\boldsymbol{r}_0 = 0 \qquad \& \qquad \mathsf{A}^{\dagger}\,\boldsymbol{l}_0 = 0 \tag{3.6a}$$

$$\mathsf{A}\,\boldsymbol{r}_i = a_i\,\boldsymbol{r}_i \qquad \& \qquad \mathsf{A}^{\dagger}\,\boldsymbol{l}_i = a_i\,\boldsymbol{l}_i \qquad\qquad i = 1, \dots, d-1 \tag{3.6b}$$

with

$$\boldsymbol{l}_i \cdot \boldsymbol{r}_j = \delta_{ij} \qquad i\,, j = 0, \dots, d-1 \tag{3.7}$$

and $\{a_i\}_{i=1}^{d-1}$ is the sequence of non-vanishing eigenvalues of A. The $\cdot$ stands for the canonical scalar product in $\mathbb{R}^d$. If the *solvability condition*

$$\boldsymbol{l}_0 \cdot (\mathsf{A}\,\boldsymbol{x}) = \boldsymbol{l}_0 \cdot \boldsymbol{f} = 0 \tag{3.8}$$

holds true then (3.1) admits a unique solution which we can write as

$$\boldsymbol{x} = \sum_{i=1}^{d-1} \frac{\boldsymbol{r}_i \otimes \boldsymbol{l}_i}{a_i} \boldsymbol{f} = \sum_{i=1}^{d-1} \frac{\boldsymbol{r}_i}{a_i} (\boldsymbol{l}_i \cdot \boldsymbol{f}) \tag{3.9}$$

3.2 Differential operators

We may regard the differential equation

$$\left(\frac{d^2}{dt^2} + \omega^2\right) x_t = f_t \tag{3.10}$$

for

$$x_t \,,\, f_t \,:\, [0, T] \mapsto \mathbb{R} \tag{3.11}$$

with x_t satisfying periodic boundary conditions

$$x_t|_{t=0} = x_t|_{t=T} \tag{3.12a}$$

$$\dot{x}_t|_{t=0} = \dot{x}_t|_{t=T} \tag{3.12b}$$

as an infinite dimensional generalization of (3.1) with the identifications

$$\mathsf{A}_{ij} \quad \sim \quad \mathfrak{A}_{tt'} := \left(\frac{d^2}{dt^2} + \omega^2\right) \delta(t - t') \tag{3.13a}$$

$$\sum_{j=1}^{d} \mathsf{A}_{ij} x_j \quad \sim \quad \int_0^T dt' \left(\frac{d^2}{dt^2} + \omega^2\right) \delta(t - t') x_{t'} = \left(\frac{d^2}{dt^2} + \omega^2\right) x_t \tag{3.13b}$$

The identification of $\mathfrak{A}$ as the kernel of a linear operator requires the specification of the functional space $\mathcal{F}$ on which the differential operation acts. By (3.12a), (3.12b) we may identify $\mathcal{F}$ with the space of differentiable periodic functions over $[0, T]$. For any $f \,,\, g \,\in\, \mathcal{F}$ we then have

$$\int_0^T dt \int_0^T dt' \, f_t \mathfrak{A}_{tt'} g_{t'} = \int_0^T dt \int_0^T dt' \, g_{t'} \mathfrak{A}_{tt'} f_t \tag{3.14}$$

meaning that $\mathfrak{A}$ is self-adjoint with respect to the standard $\mathbb{L}^2([0, T])$ scalar product:

$$\langle f \,,\, g \rangle := \int_0^T dt \, f_t g_t \tag{3.15}$$

In full analogy with the finite dimensional case the problem (3.10) admits a unique solution if

$$\mathrm{dimker}\mathfrak{A} = 0 \tag{3.16}$$

Since the functions

$$\psi_{0:0} := \sqrt{\frac{\omega}{\pi}} \cos(\omega\, t) \qquad \& \qquad \psi_{0:1} := \sqrt{\frac{\omega}{\pi}} \sin(\omega\, t) \tag{3.17}$$

are always annihilated by the differential action of $\mathfrak{A}$

$$\left(\frac{d^2}{dt^2} + \omega^2\right) \psi_{0:i} = 0 \qquad i = 0, 1 \tag{3.18}$$

we conclude that for any $n \,\in\, \mathbb{N}$

$$T = \frac{2\,\pi\, n}{\omega} \qquad \Rightarrow \qquad \mathrm{dimker}\mathfrak{A} = 2 \tag{3.19}$$

3.3 Consequences on perturbation theory

Secular terms occurs whenever the Cauchy Green functions acts on terms belonging to the kernel of the periodic differential operator:

$$\int_0^t dt' \sin[\omega\,(t-t')]\cos(\omega\,t') = \Im \int_0^t dt' \frac{e^{\imath\,\omega\,t} + e^{\imath\,\omega\,(t-2\,t')}}{2} = \frac{t\,\sin(\omega\,t)}{2} \tag{3.20}$$

We also notice that the term linear in time occurring at order $O(\varepsilon)$ effectively generates a slower time scale

$$s = \varepsilon\,t \tag{3.21}$$

We can use these observations to construct an algorithm for systematic partial resummations of the perturbative expansion.

4 Multiscale expansion and solvability condition

Let us suppose

$$x_t = \sum_{i=0}^{\infty} \varepsilon^n\,x_{(n)}(t,s,\dots) \tag{4.1}$$

with

$$s = \varepsilon\,t \tag{4.2}$$

and the $\dots$ referring to eventual time dependence upon even slower time scales $\varepsilon^n\,t$ with $n > 1$. We have then

$$\frac{d}{dt} = \partial_t + \varepsilon \partial_s + \dots \tag{4.3}$$

Partial derivatives with respect to the slower time scales do not affect the zero order of perturbation theory but intervene in the equation for the leading order correction. For the problem of section (2.1) we get into

$$\ddot{x}_{(1)} + 2\,\dot{x}'_{(0)} + \omega^2\,x_{(1)} = -\omega\,x_{(0)} \tag{4.4}$$

with

$$x'_{(0)} = \partial_s x_{(0)} \tag{4.5}$$

We now write the solution of the zeroth order equation in the form

$$x_{(0)}(t,s) = A_s \cos(\omega\,t + \phi_s) \tag{4.6}$$

where the initial conditions are now enforced by requiring

$$A_s|_{s=0} = x_o \tag{4.7a}$$

$$\phi_s|_{s=0} = 0 \tag{4.7b}$$

We now impose (4.4) the *solvability conditions*

$$\int_0^{\frac{2\,\pi}{\omega}} dt\,\psi_{0:i}(t)\,[2\,\dot{x}'_{(0)}(t,s) + \omega\,x_{(0)}(t,s)] = 0 \qquad i = 0,1 \tag{4.8}$$

to enforce orthogonality in $\mathbb{L}^2([0,T])$ sense to the kernel of the periodic differential operators acting on $x_{(1)}$. Taking into account

$$\dot{x}'_{(0)}(t,s) = -A'_s\omega\sin(\omega\, t+\phi_s) - A_s\omega\phi'_s\cos(\omega\, t+\phi_s) \tag{4.9}$$

we have

$$\begin{aligned} i=0: &\qquad 2\,\omega\,(A'_s\sin\phi_s + A_s\,\phi'_s\cos\phi_s) - \omega^2\,A_s\,\cos\phi_s = 0 \\ i=1: &\qquad 2\,\omega\,(A'_s\cos\phi_s - A_s\,\phi'_s\sin\phi_s) + \omega^2\,A_s\,\sin\phi_s = 0 \end{aligned} \tag{4.10}$$

The two equations are simultaneously satisfied if we set

$$A'_s = 0 \tag{4.11a}$$

$$\phi' = \frac{\omega}{2} \tag{4.11b}$$

whence it follows

$$x_{(0)}(t,s) = x_o\cos\left(\omega\,t + \frac{s\,\omega}{2}\right) \equiv x_o\cos\left[\omega\,t\left(1+\frac{\varepsilon}{2}\right)\right] \tag{4.12}$$

5 Anaharmonic oscillator

The equation for the anharmonic oscillator is

$$\frac{d^2x}{dt^2} + \omega^2\,x + g\,x^3 = 0 \tag{5.1}$$

We will consider the initial conditions

$$x_t|_{t=0} = x_o \qquad\qquad \dot{x}_t|_{t=0} = 0 \tag{5.2}$$

From canonical dimensional analysis

$$[\omega] = -[time] \qquad \& \qquad [g] = -2[time\times space] \tag{5.3}$$

we can define an adimensional expansion parameter if

$$\varepsilon := \frac{g\,x_o^2}{\omega^2} \ll 1 \tag{5.4}$$

5.1 Regular perturbation theory

Standard perturbation theory

$$x_t = \sum_{n=0}^{\infty} x_t^{(n)} \tag{5.5}$$

with

$$\ddot{x}_t^{(0)} + \omega^2\,x_t^{(0)} = 0 \tag{5.6a}$$

$$\ddot{x}_t^{(1)} + \omega^2\, x_t^{(1)} = -\frac{\omega^2}{x_o^2}[x_t^{(0)}]^3 \tag{5.6b}$$

$$\vdots \tag{5.6c}$$

The leading order is

$$x_t^{(0)} = x_o \cos(\omega t) \tag{5.7}$$

Energy conservation establishes a relation between the *amplitude* x_o and the energy

$$\mathcal{E} = \frac{\omega^2\, x_o^2}{2} + \frac{g\, x_o^4}{4} \tag{5.8}$$

Using the Cauchy Green function the leading order correction is

$$\begin{aligned} x_t^{(1)} &= -x_o\,\omega \int_0^t dt'\, \sin[\omega(t-t')]\, \cos^3(\omega t') \\ &= -\frac{x_o\,\omega}{8}\Im e^{\imath\,\omega\,t} \int_0^t dt' \left[e^{\imath\,2\,\omega\,t'} + 3 + e^{-\imath\,2\,\omega\,t'} + e^{-\imath\,4\,\omega\,t'}\right] \end{aligned} \tag{5.9}$$

we obtain

$$x_t^{(1)} = -\frac{3\,\omega\,t}{8} x_o \sin(\omega\,t) + \text{periodic terms} \tag{5.10}$$

Adding up the zeroth and first order we get into

$$x_t = x_o \cos(\omega\,t) - \frac{3\,\omega\,t\,\varepsilon}{8} x_o \sin(\omega\,t) + \varepsilon\,\text{periodic terms} \tag{5.11}$$

Multiscale perturbation theory justifies then the resummation

$$x_t = x_o \cos\left[\left(1 + \frac{3\,\varepsilon}{8}\right)\omega\,t\right] + \varepsilon\,\text{periodic terms} \tag{5.12}$$

Note that ε depends by definition upon x_o. This means that at variance with the linear case for which period and amplitude of a periodic orbit are independent, the period of the anharmonic oscillator *depends* upon the intensity of the amplitude x_o.

5.2 Derivation of (5.12)

The solvability conditions are now

$$\int_0^{\frac{2\,\pi}{\omega}} dt\, \psi_{0:i}(t)\, [2\,\dot{x}'_{(0)}(t,s) + \frac{\omega^2}{x_o^2}\, A_s^3 \cos^3(\omega\,t + \phi_s)] = 0 \qquad i = 0, 1 \tag{5.13}$$

after straightforward algebra they become

$$\begin{aligned} i = 0: &\qquad 2\,\omega\,(A'_s \sin\phi_s + A_s\,\phi'_s \cos\phi_s) - \frac{3\,\omega^2\,A_s^3}{4\,x_o^2}\,\cos\phi_s = 0 \\ i = 1: &\qquad 2\,\omega\,(A'_s \cos\phi_s - A_s\,\phi'_s \sin\phi_s) + \frac{3\,\omega^2\,A_s^3}{4\,x_o^2}\,\sin\phi_s = 0 \end{aligned} \tag{5.14}$$

They reduce to the simpler system

$$A'_s = 0 \qquad \Rightarrow \qquad A_s = x_o \tag{5.15a}$$

$$\phi'_s = \frac{3\,\omega}{8} \tag{5.15b}$$

whence (5.12) follows readily.

Appendix

A Green functions

A Sturm-Liouville (i.e. second order) differential operator is specified by two ingredients:

1. the differential operation on any test function f such as

$$\mathfrak{L}f = \left(\frac{d^2}{dt^2} + \omega^2\right) f \tag{A.1}$$

   in the case of the harmonic oscillator

2. the functional space $\mathcal{F}$ on which the differential operation acts.

Possible examples of $\mathcal{F}$ are

- class of differentiable functions satisfying Cauchy boundary conditions at $t = 0$

$$f(0) = \frac{df}{dt}(0) = 0 \tag{A.2}$$

- class of differentiable functions periodic in $[0, T]$:

$$f(0) = f(T) \qquad \& \qquad \frac{df}{dt}(0) = \frac{df}{dt}(T) \tag{A.3}$$

The Green function $G_{\mathcal{F}}$ yields the inverse of $\mathfrak{L}$ on a given $\mathcal{F}$:

$$\left(\frac{d^2}{dt^2} + \omega^2\right) G(t, t') = \delta(t - t') \tag{A.4}$$

This means that $G_{\mathcal{F}}$ must satisfy the boundary conditions specifying $\mathcal{F}$. Below we will drop the subscript $\mathcal{F}$ to simplify the notation. Note that (A.4) for $t \neq t'$ states that the Green function solves the homogeneous equation

$$\left(\frac{d^2}{dt^2} + \omega^2\right) G(t, t') = 0 \tag{A.5}$$

A.1 Cauchy Green function

The Cauchy Green function G satisfies for any t'

$$G(0, t') = 0 \qquad \& \qquad \frac{dG}{dt}(0, t') = 0 \tag{A.6}$$

The explicit form of G is found by splitting $[0, T]$ in two sub-interval.

- For $t < t'$ the boundary conditions (A.6) imply that (A.5) is satisfied by setting

$$G(t, t') = 0 \tag{A.7}$$

- The solution for $t > t'$ must satisfy the boundary condition imposed by the Dirac-δ at t':

$$\int_{s-\epsilon}^{s+\epsilon} dt \left(\frac{d^2}{dt^2} + \omega^2\right) G(t,t') = \frac{d}{dt} G(t,t')|_{t'-\epsilon}^{t'+\varepsilon} = \int_{s-\epsilon}^{s+\varepsilon} dt\, \delta(t-t') = 1 \tag{A.8}$$

whence

$$\lim_{t \searrow t'} \frac{dG}{dt}(t,t') = 1 \tag{A.9}$$

while we can still require continuity at $t = t'$:

$$\lim_{t \searrow t'} G(t,t') = 1 \tag{A.10}$$

Gleaning the above information we get into

$$G(t,s) = \begin{cases} \dfrac{\sin[\omega(t-s)]}{\omega} & \text{if} \quad t \geq s \\ 0 & \text{if} \quad t < s \end{cases} = \frac{\sin[\omega(t-s)]}{\omega} H_{[0]}(t-s) \tag{A.11}$$

A.2 Periodic Green function

Let us now construct the Green function of the harmonic oscillator with periodic boundary conditions in $[0,T]$. In the two sub-intervals determined by t' the Green function is solution of the homogeneous equation (A.5)

$$G(t,t') = \begin{cases} A\, \cos(\omega t + \phi) & t \leq t' \\ A'\, \cos(\omega t + \phi') & t > t' \end{cases} \tag{A.12}$$

Periodicity imposes the conditions

$$A\, \cos\phi = A'\, \cos(\omega T + \phi') \tag{A.13a}$$

$$A\, \sin\phi = A'\, \sin(\omega T + \phi') \tag{A.13b}$$

satisfied by setting

$$\phi = \phi' + \omega\, T \qquad \& \qquad A = A' \tag{A.14}$$

A further condition comes from the discontinuity of the derivative of the Green function at $t = t'$

$$A' \sin\left(\omega\, t' + \phi'\right) - A\, \sin\left(\omega\, t' + \phi\right) = -\frac{1}{\omega} \tag{A.15}$$

Upon inserting (A.14) into (A.15), the choice

$$\phi = -\omega\, t' + \frac{\omega\, T}{2} \tag{A.16}$$

yields

$$A = \frac{1}{2\,\omega\, \sin\frac{\omega T}{2}} \tag{A.17}$$

The periodic Green function is therefore

$$G(t,t') = \begin{cases} \dfrac{\cos\left[\omega\,(t-t') + \frac{\omega\,T}{2}\right]}{2\,\omega\,\sin\frac{\omega\,T}{2}} & t\,\leq\,t' \\ \dfrac{\cos\left[\omega\,(t-t') - \frac{\omega\,T}{2}\right]}{2\,\omega\,\sin\frac{\omega\,T}{2}} & t\,>\,t' \end{cases} = \frac{\cos\left[\omega\,|t-t'| - \frac{\omega\,T}{2}\right]}{2\,\omega\,\sin\frac{\omega\,T}{2}} \tag{A.18}$$

The Green function does not exist for

$$\frac{\omega\,T}{2} \in \pi\,\mathbb{N} \tag{A.19}$$

The reason is that whenever (A.19) is satisfied

$$\psi_{0:0} = \sqrt{\frac{\omega}{\pi}}\cos(\omega t) \tag{A.20a}$$

$$\psi_{0:1} = \sqrt{\frac{\omega}{\pi}}\sin(\omega t) \tag{A.20b}$$

are periodic elements of the kernel of (A.4). It is interesting to compute the projection of the periodic Green function onto the complement of its null-space. To this goal we can set

$$\bar{\omega} = \frac{2\,\pi}{T} \tag{A.21}$$

and compute e.g.

$$\int_0^T ds\,\frac{\cos\left[\omega\,|t-t'| - \frac{\omega\,T}{2}\right]}{2\,\omega\,\sin\frac{\omega\,T}{2}}\cos\bar{\omega}s = \frac{\cos\bar{\omega}t}{\omega^2-\bar{\omega}^2} \tag{A.22}$$

and

$$\int_0^T ds\,\frac{\cos\left[\omega\,|t-t'| - \frac{\omega\,T}{2}\right]}{2\,\omega\,\sin\frac{\omega\,T}{2}}\sin\bar{\omega}s = \frac{\sin\bar{\omega}t}{\omega^2-\bar{\omega}^2} \tag{A.23}$$

Thus the candidate Green function on the orthogonal complement of the null-space is

$$\tilde{G}_{\perp}(t,s) = \frac{\cos\left[\omega\,|t-t'| - \frac{\omega\,T}{2}\right]}{2\,\omega\,\sin\frac{\omega\,T}{2}} - \frac{\bar{\omega}}{\pi}\frac{\cos\bar{\omega}t\cos\bar{\omega}t' + \sin\bar{\omega}t\sin\bar{\omega}t'}{\omega^2-\bar{\omega}^2} \tag{A.24}$$

having taken into account the normalization of the eigenfunctions. We can now isolate the addends diverging for $\bar{\omega}\to\omega$

$$\tilde{G}_{\perp}(t,s) = \frac{\sin\omega\,|t-t'|}{2\,\omega} + \frac{\cos\omega\,|t-t'|}{2\,\omega\,\sin\frac{(\omega-\bar{\omega})\,T}{2}} - \frac{\bar{\omega}}{\pi}\frac{\cos\bar{\omega}(t-t')}{\omega^2-\bar{\omega}^2} \tag{A.25}$$

whence finally

$$G_{\perp}(t,s) = \lim_{\bar{\omega}\to\omega}\tilde{G}_{\perp}(t,t') = \frac{\sin\omega\,|t-t'|}{2\,\omega} \tag{A.26}$$

Acting on functions orthogonal to the kernel $G_{\perp}(t,t')$ reduces to the Cauchy Green function

$$\int_0^T ds\,G_{\perp}(t,s)\,f(s) = \int_0^t dt'\frac{\sin\omega(t-t')}{\omega}f(t') - \int_0^T dt'\frac{\sin\omega(t-t')}{2\,\omega}f(t') = \int_0^t dt'\frac{\sin\omega(t-t')}{\omega}f(t') \tag{A.27}$$

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