# Chapter 2

# Homological algebra

## 2.1 Boundary operator

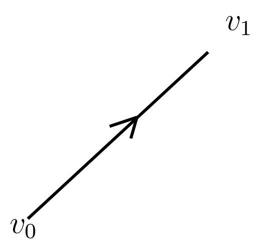
It is now time to put some algebra in algebraic topology.
Consider a (geometric) simplex $[v_0, \ldots, v_n]$ of some $\Delta$-complex $K$ (so that some faces might be glued together). We would like to consider its boundary as some sort of the " sum " of the $(n-1)$-dimensional simplices it is made of. Of course this boundary as a set is just the union

$$\cup_{i=0}^{n}[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$$

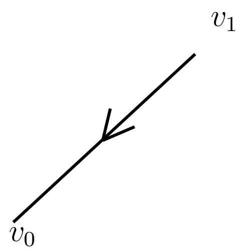
and every $(n-1)$-dimensional face has its natural order, but this order might not "suit" the orientation of the whole boundary/simplex itself. To see what this means we must discuss the concept of the orientation of a simplex. Let us first consider the cases $n = 1, 2, 3$.

It is easy to decide what is meant by the orientation of a 1-simplex $[v_0, v_1]$ - it is what we defined as "order". We can think of a this simplex as a directed "arrow", a path from the point $v_0$ to the point $v_1$. These points together form the boundary of this path. If we change this direction to its opposite, we get an arrow $[v_1, v_0]$ which goes from $v_1$ to $v_0$. Hence it is natural to think that $[v_1, v_0]$ is $[v_0, v_1]$ "with an opposite sign", hence we write

$$[v_1, v_0] = -[v_0, v_1].$$




simplex $[v_0, v_1]$

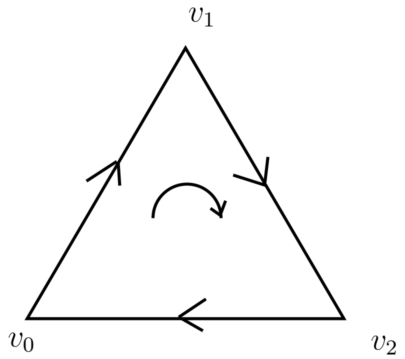



simplex $[v_1, v_0]$

Hence 1-simplex has two orientations. Interchange of the order of its vertices switches the orientation to the opposite orientation.

Next we consider a triangle $[v_0, v_1, v_2]$.

Its boundary is a "continuous" (meaning here "connected") closed path from $v_0$ to itself, going through $v_1$ and $v_2$. It has a natural orientation (see the picture below) - first one goes from $v_0$ to $v_1$, i.e. "travels" the edge $[v_0, v_1]$, then from $v_1$ to $v_2$, i.e. "travels" the edge $[v_1, v_2]$, and finally the edge $[v_2, v_0]$. This also defines a natural orientation of the whole triangle (indicated in the picture by the arc-shaped arrow in the centre of the triangle) - which is " clockwise " orientation in the case of this particular triangle.




Hence if we think of the boundary $\partial[v_0, v_1, v_2]$ as a geometrical entity we might want to write something like

$$\partial[v_0, v_1, v_2] = [v_0, v_1] + [v_1, v_2] + [v_2, v_0].$$

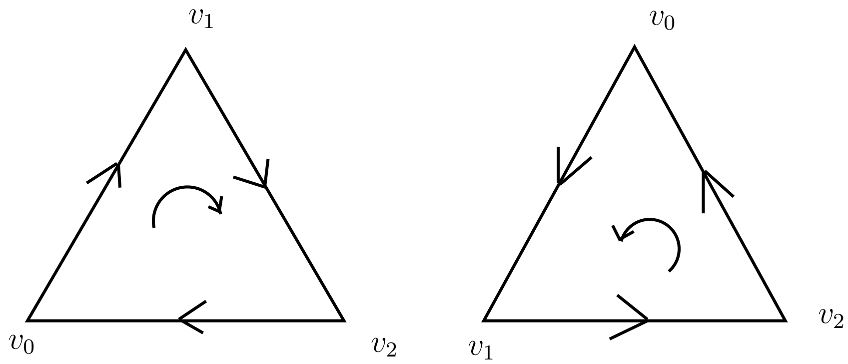
This makes geometrical sence, as explained above, but we see that when we think about the matter in this way faces $[v_0, v_1]$ and $[v_1, v_2]$ preserve their natural ordering inhereted from $[v_0, v_1, v_2]$, while the 3rd face $[v_2, v_0]$ has an opposite direction! If we remember our previous agreement that

$$[v_2, v_0] = -[v_0, v_2],$$

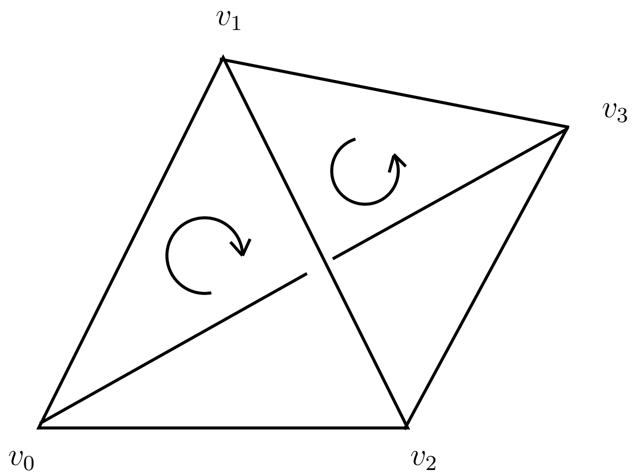
we obtain "the formula"

$$\partial[v_0, v_1, v_2] = [v_0, v_1] + [v_1, v_2] - [v_0, v_2].$$

Going back to the orientation questions, we now have intuitive idea of the two possible natural ways to give the orientation to the boundary of the triangle and hence the triangle itself - it is a clockwise way to travel a circle (the boundary) and the counterwise way. If we interchange two vertices - say $v_0$ and $v_1$, the orientation switches to the opposite, as the picture below shows. Hence a 2-simplex has two orientations and they also have the property of switching to the opposite, when two vertices are interchanged (Exercise 2.1).




Before going to the general formalities let us check the last case we can draw - the 3-simplex $[v_0, v_1, v_2, v_3]$ i.e. a tetrahedron. Now its 2-faces are triangles, and we already know what we mean by the orientation ("clockwise" or "counterclockwise") for triangles.




We see immediately that faces $[v_0, v_1, v_2]$ and $[v_1, v_2, v_3]$ have opposite orientations. One can check all the triangles and compare their orientations (exercise 2.2) in the same fashion. As a result one obtains that the faces $[v_1, v_2, v_3]$ (0-face) and $[v_0, v_1, v_3]$ (2-face) have the same orientation, while $[v_0, v_2, v_3]$ (1-face)and $[v_0, v_1, v_2]$ (3-face)also have the same orientation - opposite to the orientation of $[v_1, v_2, v_3]$ and $[v_0, v_1, v_3]$. If we call the first orientation "positive" and the other one " negative", we obtain the formula

$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2].$$

It is now clear that the combination of these orientations of the triangles in the boundary define what we should think of as an orientation of the whole tetrahedron, although it is more difficult to give it a simple geometrical interpretation as we did in the cases $n = 1$ and $n = 2$. Let us still observe what will happen if we interchange the order of two vertices, $v_0$ and $v_1$. There are two faces, the 2-face and the 3-face that contain both vertices, so in the interchange their orientation switches to the opposite. What about 0-face and 1-face? Now the orientation of the 0-face does not change, since the order of its vertices remain the same. But in the "new" ordered simplex $[v_1, v_0, v_2, v_3]$ it is not 0-face anymore, it is 1-face, so its orientation is switched as well. The same is true for the face $[v_1, v_2, v_3]$.
Hence the interchange of two vertices switches the orientation of all the faces, in least in the case of vertices $v_0$ and $v_1$. The reader is invited to check all the other cases as an exercise.

We are now ready to formalize these observations.
Recall that a **permutation** of the finite set $\{0, \dots, n\}$ is a bijection $\alpha\colon \{0, \dots, n\} \to \{0, \dots, n\}$. If $v_0 < v_1 < \dots < v_n$ is a particular ordering of the vertices of a simplex and $\alpha$ is a permutation of the set $\{0, \dots, n\}$ one can define another ordering by

$$v_{\alpha(0)} < v_{\alpha(1)} < \dots < v_{\alpha(n)}.$$

Also we can obtain **any** other ordering $v'_0 < v'_1 < \dots < v'_n$ of the same set of vertices from the ordering $v_0 < v_1 < \dots < v_n$ in this way with a unique permutation $\alpha$ - it is the only permutation that maps $i$ to $j = \alpha(i)$ with $v'_i = v_j$.
Recall that a permutation $\alpha$ of the set $\{0, \dots, n\}$ is called a **transpose** if it interchanges two elements leaving other elements fixed. To be more precise there exist $i, j \in \{0, \dots, n\}$, $i \neq j$ so that $\alpha(i) = j$, $\alpha(j) = i$ and $\alpha(k) = k$ if $k \neq i, j$. In this case one often writes $\alpha = (ij)$.
Every permutation of the set $\{0, \dots, n\}$ can be written as a composition of transposes. This representation is not unique, but the oddity of the amount of transposes needed to represent a given permutation **is** an invariant of the permutation. In other words if $\alpha$ is a permutation that can be written as a composition of $n$ transposes as well as the composition of $m$ transposes, both $n$ and $m$ are even or both are odd. In the former case permutation is called **even**, in the latter case - **odd**. This fact is proved in the Linear Algebra I

course.

Now suppose $v_0 < v_1 < \ldots < v_n$ and $v'_0 < v'_1 < \ldots < v'_n$ are two different orderings of the same set of vertices of a simplex $\sigma$. Let $\alpha$ be the unique permutation of the set $\{0, \ldots, n\}$ for which $v'_i = v_{\alpha(i)}$.
We say that ordered simplices $[v_0, \ldots, v_n]$ and $[v'_0, \ldots, v'_n]$ **have the same orientation** (or **oriented coherently**) if $\alpha$ is even. If $\alpha$ is odd we say that they have **opposite orientation**.
The relation "$[v_0, \ldots, v_n]$ and $[v'_0, \ldots, v'_n]$ have the same orientation" is an equivalance relation in the set of all orderings of the set $\{v_0, \ldots, v_n\}$.
**The orientation** is an equivalence class of this relation. Every simplex has two orientations, except if it a 0-simplex. A 0-simplex has only one orientation.

These definitions come from our observation that transposes (which are odd permutations) must switch the orientation to the opposite. It is also natural to think that the composition of permutations is compatible with the orientation switching. Since all permutations can be written as a composition, these two natural requirements define the notion of orientation uniquely.

Suppose $[v_0, \ldots, v_n]$ is an ordered simplex. Consider its $i$th face $\sigma_i = [v_0, \ldots, \widehat{v_i}, \ldots, v_n]$ and its $j$th face $\sigma_j = [v_0, \ldots, \widehat{v_j}, \ldots, v_n]$, $i < j$. Now we can think of $\sigma_i$ and $\sigma_j$ as "models" for the same $n-1$-simplex, with the same vertices $v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_n$ in the same order and the last vertex, which is labelled $v_j$ in $\sigma_i$ and $v_i$ in $\sigma_j$. Let us call this vertex $b$. Then $\sigma_i$ and $\sigma_j$ are identified as "the same " simplex but with different ordering of vertices. To get from the ordering $[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, b, v_{j+1}, \ldots, v_n]$ to the ordering $[v_0, \ldots, v_{i-1}, b, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n]$ one needs $j - i$ transposes - first you interchange $b$ with $v_{j-1}$, then $b$ with $v_{j-2}$ and so on, until $b$ riches the $i$th place in the ordering.
Hence the orientations of $\sigma_i$ and $\sigma_j$ are coherent if $j - i$ is even and opposite if $j - i$ is odd.
As a conclusion we see that $(n-1)$-faces with even indexes all have the same orientation, while faces with odd indexes also have the same orientation, opposite to the orientation of even-indexed faces. If we call the former orientation positive and the latter - negative, we obtain the formula

$$\partial[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \widehat{v_i}, \ldots, v_n].$$

The only problem that remains at this point is to give this formula a formal algebraic meaning. We added and substracted simplices together, what

does it mean? We didn't define any addition operation on the set of simplices!
It turns out we don't really need to, since every set can be "extended" to an abelian group in a natural and universal way. This is what free abelian groups are all about.

The idea of the construction is the following. Suppose a set $A$ is "imbedded" in an abelian group $G$. Since $G$ contains all elements $a \in A$ it also contains all integer multiplies $na$, $n \in \mathbb{Z}, a \in A$. Again $G$ must contain all the possible finite sums of these elements

$$n_1a_1 + \ldots + n_ka_k,$$

where $n_i \in \mathbb{Z}$, $a_i \in A$. We can refer to such sums as " formal sums" of elements of $A$. Every formal sum can be identified with the indexed collection of the integer coefficients $(n_1, \ldots, n_k)$. For the element $a$ of $A$ which does not occur in the sum above (i.e. $a \neq a_1, \ldots, a_n$) we can think that it does occur with the coefficient $n_a = 0$. In this manner we can extend the indexed family $(n_1, \ldots, n_k)$ to the indexed family $(n_a)_{a \in A}$. Such an indexed family can be thought of as an element of $\mathbb{Z}^A$ i.e. a function $f\colon A \to \mathbb{Z}$. Moreover every indexed family that comes from a formal sum has the property that only finite amount of indexes differ from the zero.

**Definition 2.1.1.** *A function $f\colon A \to \mathbb{Z}$ is said to be* **finetely supported** *if*

$$B_f = \{a \in A \mid f(a) \neq 0\}$$

*is a finite subset of $A$. This set is called* **the carrier** *of $f$.*

The subset of $\mathbb{Z}^A$ consisting of the finitely supported functions is denoted $\mathbb{Z}^{(A)}$. It is clear that the element $f \in \mathbb{Z}^A$ is finitely supported if and only if there exists a finite $B \subset A$ such that $f(a) = 0$ for all $a \notin B$.
The set $\mathbb{Z}^A$ has a natural structure of an abelian group, with addition defined "pointwise",

$$(f + g)(a) = f(a) + g(a) \text{ for all } a \in A.$$

The neutral element is the constant zero function $0\colon A \to \mathbb{Z}$ and the inverse element of $f$ is $-f$ defined pointwise by

$$(-f)(a) = -f(a), a \in A.$$

**Lemma 2.1.2.** *$\mathbb{Z}^{(A)}$ is a subgroup of $\mathbb{Z}^A$.*

*Proof.* Exercise 2.3. ∎

Of course if $A$ is finite $\mathbb{Z}^{(A)} = \mathbb{Z}^A$.
If $A = \{a\}$ is a singleton, we also write $\mathbb{Z}^{\{a\}} = \mathbb{Z}(a)$. Hence if $A = \{a_1, \ldots, a_n\}$ is finite, we have

$$\mathbb{Z}^{(A)} = \mathbb{Z}(a_1) \oplus \mathbb{Z}(a_2) \oplus \ldots \oplus \mathbb{Z}(a_n).$$

**Definition 2.1.3.** *Suppose $G$ is an abelian group. A subset $A \subset G$ is called* **independent** *if for all $a_k \in A, k = 1, \ldots, n$ and $n_1, \ldots, n_k \in \mathbb{Z}$ the condition*

$$n_1 a_1 + \ldots + n_k a_k = 0$$

*is equivalent to $n_1 = \ldots = n_k = 0$. If independent subset $A$ also* **generates** *the whole group i.e. every element $x \in G$ can be written as a finite sum*

$$x = n_1 a_1 + \ldots + n_k a_k,$$

*where $n_i \in \mathbb{Z}, a_i \in A$, we say that $A$ is* **a basis** *of $G$.*

It follows straight from the definition that a subset $A \subset G$ is a basis of $G$ if and only if every element $x \in G$ has a **unique** representation as a formal sum

$$\sum_{a \in A} n_a a,$$

where the family $(n_a)_{a \in A}$ is finitely supported.

Basis have the following important extension property.

**Lemma 2.1.4.** *Suppose $G$ is an abelian group and a subset $A \subset G$ is a basis of $G$. Suppose $f \colon A \to H$, where $H$ is an abelian group, is a mapping of sets. Then there exists the unique group homomorphism $g \colon G \to H$ which is an extension of $f$ i.e. $g(a) = f(a)$ for all $a \in A$.*

*Proof.* Exercise 2.4. □

Let us construct a canonical basis for $\mathbb{Z}^{(A)}$. For every $a \in A$ let

$$f_a \in \mathbb{Z}^{(A)}$$

be defined by

$$f_a(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise} . \end{cases}$$

It is clear that $f_a \neq f_b$ if $a \neq b$, so we identify $a$ with $f_a$ and think of $A$ as a subset $\{f_a \mid a \in A\}$ of $\mathbb{Z}^{(A)}$.

**Lemma 2.1.5.** *The set $\{f_a \mid a \in A\}$ is a basis of the abelian group $\mathbb{Z}^{(A)}$.*

*Proof.* Exercise 2.5. □

Of course not all abelian groups have basis. For example $\mathbb{Z}_n$ or $\mathbb{Q}$ do not have any basis (exercises 2.7 and 2.8).
The equivalent way to describe the groups that do have a basis is the notion of the free group.

**Definition 2.1.6.** *Suppose $A$ is a set. A pair $(F_A, i)$ , where $F_A$ is an abelian group and $i\colon A \to F_A$ is a mapping, is called* **a free abelian group on the set** *$A$, if it satisfies the following universal property.*

*Suppose $f\colon A \to G$ is a mapping, where $G$ is an abelian group. Then there exists a unique group homomorphism $g\colon F_A \to G$ such that $g \circ i = f$.*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & G \\ & F_A \arrow[ur, "g"'] & \end{tikzcd}$$

*An abelian group $G$ is called* **free** *if there exists a set $A$ and $i\colon A \to G$ such that $(G, i)$ is a free group on the set $A$.*

Of course free group on the set $A$ is not "unique" - any isomorphic group will also do the trick. But this is the worst that can happen.

**Lemma 2.1.7.** *Suppose $A$ is a set and $(F_A, i)$ as well as $(F'_A, i')$ are both free abelian groups on the set $A$. Then there exists (unique) group isomorphism $g\colon F_A \to F'_A$ such that $g \circ i = i'$.*

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ F_A \arrow[rr, "g"] & & F'_A \end{tikzcd}$$

*Proof.* Exercise 2.6. □

This handles uniqueness. What about existence? Well, in fact we already constucted an example of a free group for any set $A$, so it only remains to prove it satisfies the definition.

**Proposition 2.1.8.** *Suppose $A$ is a set, $G$ is an abelian group and $j\colon A \to G$ is a mapping. Suppose $j$ is injective and $j(A)$ is a basis of $G$. Then $(G, j)$ is a free abelian group on the set $A$*

*Proof.* Suppose $f\colon A \to H$ is a mapping, where $H$ is an abelian group.
Since $j$ is an injection we can define the mapping $f'\colon j(A) \to H$ by $f'(j(a)) = f(a)$.
Since $\{j(A) \mid a \in A\}$ is a basis of $G$, we can extend this mapping to a unique homomorphism $g\colon G \to H$ (Lemma 2.1.4). This mapping has the property $g \circ j = f$ by constuction. Moreover any homomorphism $g'\colon G \to H$ with the same property must be $g$, by the uniqueness of the extension. □

**Proposition 2.1.9.** *Let $A$ be a set. Define $i\colon A \to \mathbb{Z}^{(A)}$ by $i(a) = f_a$. Then $(\mathbb{Z}^{(A)}, i)$ is a free abelian group on the set $A$.*

*Proof.* This follows from the previous proposition, since $i$ in injective and

$$i(A) = \{f_a : a \in A\}$$

is a basis of $\mathbb{Z}^{(A)}$ by the Lemma 2.1.5. □

By the 2.1.7 it follows now that every free group on the set $A$ is isomorphic to $\mathbb{Z}^{(A)}$. Using this it is easy to prove the following result.

**Proposition 2.1.10.** *Suppose $G$ is a group, $A$ is a set and $j\colon A \to G$ is a mapping. Then $(G, j)$ is free abelian on the set $A$ if and only if $j$ is injective and $j(A)$ is a basis for $G$.*
*An abelian group is free if and only if it has a basis.*

*Proof.* If $(G, j)$ is free abelian, by 2.1.7 there exists an isomorphism $g\colon \mathbb{Z}^{(A)} \to G$ such that $g \circ i = j$. Since $i(A)$ is a basis for $\mathbb{Z}^{(A)}$ and isomorphisms clearly preserve bases, it follows that $j(A) = g(i(A))$ is a basis for $G$. Moreover $j = g \circ i$ is an injection as a composition of injections.

Converse statement is proved in the proposition 2.1.8. □

If $A$ is a basis of an abelian group $G$ we also call elements of $A$ **free generators** of $G$.
It can be shown that the size of the basis determine the free group uniquely up to an isomorphism, i.e. $\mathbb{Z}^{(A)} \cong \mathbb{Z}^{(B)}$ if and only if there is a bijection between the sets $A$ and $B$. In the exercise 2.9 you are asked to prove this in the case at least one of the sets $A$ or $B$ is finite, which will be enough for our purposes.

Returning to our main course, consider a $\Delta$-complex $K$. For every $n \in \mathbb{N}$ denote by $K_n$ a collection of geometric $n$-simplices of $K$ (i.e. two simplices are considered the same if they are identified in $|K|$).
Define $C_n(K)$ to be the free abelian group on the set $K_n$. We identify corresponding element of basis with an element of $K_n$, thus elements of $C_n(K)$

are formal sums of geometric $n$-simplices with integer coefficients. The group $C_n(K)$ is called **the group of simplicial $n$-chains of the complex $K$**.

**Example 2.1.11.** *Consider a $\Delta$-complex $K(\sigma)$ where $\sigma$ is an ordered 2-simplex $[v_0, v_1, v_2]$. It consists of all the faces of $\sigma$, with no identifications. Now $C_n(K)$ is zero for $n > 2$, since complex do not have simplices in these dimensions. For $n = 2$ there is only one 2-simplex, so $C_2(K)$ is a free group based on one element $[v_0, v_1, v_2]$, hence isomorphic with $\mathbb{Z}$. Its elements have the form $n[v_0, v_1, v_2]$, $n \in \mathbb{Z}$.*
*For $n = 1$ there are three 1-simplices, so $C_1(K)$ is a free group on 3 free generators, isomorphic to $\mathbb{Z}^{(3)}$. Elements can be written uniquely in the form*

$$n[v_0, v_1] + m[v_0, v_2] + l[v_1, v_2], n, m, l \in \mathbb{Z}.$$

*Since there are 3 vertices, $C_0(K)$ is also free on 3 generators, with elements of the form*

$$nv_0 + mv_1 + lv_2$$

*(we write $[v_0] = v_0$ to simplify the notation).*

*If we identify all vertices of $\sigma$ we obtain another $\Delta$-complex $K'$. This has the same groups $C_n(K')$ as $C_n(K)$ for $n \neq 0$, but $C_0(K)$ is free on one element $v_0 = v_1 = v_2$, since all the vertices are the same now.*
*If we identify two 1-sides $[v_0, v_1]$ and $[v_1, v_2]$ we obtain another $\Delta$-complex $K''$, for which $C_2(K'')$ is free on two elements - one being $[v_0, v_2]$ and the other being $[v_0, v_1] = [v_1, v_2]$.*

Let us constuct another extremely important example.
Suppose $X$ is a topological space. For every $n \in \mathbb{N}$ let

$$\mathrm{Sing}_n(X) = \{f\colon \Delta^n \to X \mid f \text{ is continuous } \}.$$

Here $\Delta_n$ is a canonical $n$-simplex defined by

$$\Delta_n = \{(x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i \leq 1\}.$$

We consider $\Delta_n$ as an ordered simplex $(e_0, e_1, \ldots, e_n)$, where $e_0 = 0$ and $e_i$ is an $i$th element of the standard basis of $\mathbb{R}^n$.

Elements of $\mathrm{Sing}_n(X)$ are called **the singular $n$-simplices in $X$**.

**Definition 2.1.12.** *Suppose $X$ is a topological space. For every $n \geq 0$ we define $C_n(X)$ to be the free abelian group with basis* $\mathrm{Sing}_n(X)$. *Elements of $C_n(X)$ are called* **singular $n$-chains in $X$**.

If $K$ is a $\Delta$-complex we have the group of simplicial $n$-chains $C_n(K)$ and the group of singular $n$-chains $C_n(|K|)$. There is a natural way to consider $C_n(K)$ as a subgroup of $C_n(|K|)$.
Suppose $\sigma = [v_0, \ldots, v_n]$ is a geometric $n$-simplex of $K$. Then we define $i(\sigma) = f_\sigma \in C_n(|K|)$ to be a characteristic mapping of $\sigma$. Different geometric simplices define different characteristic mappings, hence $i$ is an injection, that maps a generator of $C_n(K)$ to a generator of $C_n(|K|)$. It follows that $i$ defines (by the universal property of a free group) an injective homomorphism $C_n(K) \to C_n(|K|)$. Hence we can identify $\sigma = [v_0, \ldots, v_n]$ with a corresponding characteristic mapping and we will regard $C_n(K)$ a subgroup of $C_n(|K|)$.

Next we define a boundary operator on $C_n(X)$.
Suppose $i \in \{0, \ldots, n\}$. There is a unique order-preserving simplicial mapping $\varepsilon_n^i \colon \Delta^{n-1} \to \Delta^n$ defined by

$$\varepsilon_n^i(e_j^{n-1}) = e_j, \text{ if } j < i,$$

$$\varepsilon_n^i(e_j^{n-1}) = e_{j+1}, \text{ if } j \geq i,$$

Hence $\varepsilon_n^i$ is the unique order-opreserving simplicial mapping $\Delta^{n-1} \to \Delta^n$ whose image is the $i$th face $[e_0, \ldots, \hat{e_i}, \ldots, e_n]$ of $\Delta^n$.
For a generator $f \colon \Delta^n \to X$ of $C_n(X)$ we define

$$\partial_n^i(f) = f \circ \varepsilon_n^i \colon \Delta^{n-1} \to X.$$

Mapping $\partial_n^i(f)$ is evidently continuous, hence a (generator) element of $C_{n-1}(X)$. We call it **the $i$th face of the singular simplex $f$**.

Now for the generator $f \in \mathrm{Sing}_n(X)$ define its boundary by the formula

$$\partial f = \sum_{i=0}^{n} (-1)^i \partial_n^i(f) \in C_{n-1}(X),$$

which is inspired by the considerations in the beginning of this section.
By the universal property of the free groups, we can extend $\partial$ to a homomorphism $\partial = \partial_n \colon C_n(X) \to C_{n-1}(X)$ in a unique way. This homomorphism is called **a boundary operator**.

Suppose $X = |K|$, where $K$ is a $\Delta$-complex. Consider $C_n(K)$ as a subgroup of $C_n(|K|)$ as above. Then for the generator $[v_0, \ldots, v_n] \in C_n(K)$ we have

$$\partial[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n] \in C_{n-1}(K),$$

hence $\partial$ restricted to $C_n(K)$ maps into $C_{n-1}(K)$.

The basic property of the boundary operator and the starting point for the homological methods is the following theorem.

**Theorem 2.1.13.** *For all $n \geq 2$*

$$\partial_{n-1} \circ \partial_n = 0.$$

Let us first prove the following technical result

**Lemma 2.1.14.** *Suppose $n > 1$ and $0 \leq j < i \leq n$. Then*

$$\partial^j_{n-1}(\partial^i_n f) = \partial^{i-1}_{n-1}(\partial^j_n f)$$

*for all $f \in \mathrm{Sing}_n(X)$.*

*Proof.* Exercise 2.10. □

**Prove of the theorem** 2.1.13:

*Proof.* Let $f \in \mathrm{Sing}_n(X)$. Then

$$\partial_n f = \sum_{i=0}^{n} (-1)^i \partial^i_n(f),$$

hence

$$\partial_{n-1}\partial_n(f) = \sum_{i=0}^{n} (-1)^i \partial_{n-1}\partial_n(f) =$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^i (-1)^j \partial^j_{n-1}\partial^i_n(f) = A + B,$$

where

$$A = \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} \partial^j_{n-1}\partial^i_n(f),$$

$$B = \sum_{0 \leq j < i \leq n} (-1)^{i+j} \partial_{n-1}^{j} \partial_{n}^{i}(f).$$

The change of index $i$ to $k = i - 1$ in the last sum shows that we can also write

$$\begin{aligned} B &= \sum_{0 \leq j \leq k \leq n-1} (-1)^{k+j+1} \partial_{n-1}^{j} \partial_{n}^{k+1}(f) = \\ &= - \sum_{0 \leq j \leq k \leq n-1} (-1)^{k+j} \partial_{n-1}^{k} (\partial_{n}^{j} f) = -A \end{aligned}$$

where the previous lemma is used in the second to last equation. Hence $A + B = 0$ and the claim is proved for the free generators. This suffices. □

Theorem 2.1.13 shows that singular chain groups (as well as simplicial chain groups) form an example of what is generally known as a chain complex.

**Definition 2.1.15. A chain complex** $(C, \partial)$ *is a collection* $(C_n)_{n \in \mathbb{Z}}$ *of abelian groups indexed on the set* $\mathbb{Z}$*, together with the collection of homomorphisms* $\partial_n \colon C_n \to C_{n-1}$ *(called* **boundary homomorphisms***) defined for every* $n \in \mathbb{Z}$ *such that*

$$\partial_{n-1} \circ \partial_n \colon C_n \to C_{n-2}$$

*is a zero homomorphism.*

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots$$

*If* $C_n = 0$ *for* $n < 0$ *a chain complex is said to be* **non-negative***. If all groups* $C_n$ *are free abelian groups, the complex* $C$ *is said to be* **free***.*

Theorem 2.1.13 shows that the groups $C_n(X)$ together with boundary homomorphisms $\partial_n$ defined above form a chain complex. To be precise we defined these groups only for $n \geq 0$ (and $\partial_n$ only for $n > 0$). We extend the definition by asserting $C_n(X) = 0$ for $n < 0$ and $\partial_n = 0$ for $n \geq 0$. Clearly equation $\partial_{n-1} \circ \partial_n = 0$ is then satisfied also for $n \leq 1$. Hence we obtain a non-negative free chain complex $C(X)$, called **the singular chain complex of the topological space** $X$.

If $X = |K|$, $K$ a $\Delta$-complex, the subgroups $C_n(K)$ equipped with the restrictions of the boundary homomorphisms also form a non-negative free chain complex. This complex is called **the simplicial chain complex of the $\Delta$-complex $K$**.

Now suppose $(C, \partial)$ is an arbitrary chain complex. Denote

$$Z_n(C) = \operatorname{Ker} \partial_n,$$

$$B_n(C) = \operatorname{Im} \partial_{n+1}.$$

Both $Z_n(C)$ and $B_n(C)$ are subgroups of $C_n$. Elements of $Z_n(C)$ are called **$n$-cycles of the complex $C$**, elements of $B_n(C)$ are called **$n$-boundaries of the complex $C$**.
Suppose $x \in B_n(C)$. Then $x = \partial_{n+1} y$ for some $y \in C_{n+1}$, hence

$$\partial_n(x) = \partial_n \partial_{n+1} y = 0.$$

This implies that $x \in Z_n(C)$. We showed that

$$B_n(C) \subset Z_n(C),$$

i.e. $B_n(C)$ is a subgroup of $Z_n(C)$. Since our groups are abelian, all subgroups are automatically normal, so we can form a quotient group

$$H_n(C) = Z_n(C)/B_n(C).$$

This group is called **the $n$-th homology group** of the chain complex $C$. The elements of $H_n(C)$ are equivalence classes of the $n$-cycles $x \in Z_n(C)$, denoted $[x] = x + B_n(C) \in H_n(X)$. Two elements $x, y$ of $Z_n(C)$ -define the same homology class if and only if $x - y \in B_n(C)$ i.e. $x - y = \partial_{n+1} z$ for some $z \in C_{n+1}$.

By applying this construction to the singular chain complex $C(X)$ we obtain for every $n \in \mathbb{N}$ the homology group $H_n(C(X))$, which will be denoted simply by $H_n(X)$ and called **the $n$-th singular homology group of the topological space $X$**.
Likewise for a $\Delta$-complex $K$ we obtain for every $n \in \mathbb{Z}$ the homology group $H_n(C(K))$ of the simplicial chain complex $C(K)$, which is denoted simply by $H_n(K)$ and called **the $n$-th simplicial homology group of the $\Delta$-complex $K$**.
Of course in both cases trivially $H_n(X) = 0 = H_n(K)$ for $n < 0$, since complexes are non-negative, so only non-negative dimensional homology groups are interesting.

At this point this seems like a purely abstract mathematical game. Surely we can define quotient groups $Z_n(X)/B_n(X)$, but why should we?

To give a little bit of a geometrical motivation consider a boundary of a 2 simplex which algebraically is

$$[v_0, v_1] + [v_1, v_2] - [v_0, v_2]$$

and topologically is a sphere $S^1$. Now if you consider its image in some space $X$, corresponding singular chain is a cycle, since already as a subset of a 2-simplex it is the boundary $\partial[v_0, v_1, v_2]$, so its own boundary is zero by the basic property of the boundary operator. Hence it defines a class in the homology group $H_1(X)$.
Geometrically this image looks like a sphere $S^1$, in other words "it looks like a (1-dimensional) hole ". Now if we can "fill this hole", in other words if we can find the image of the 3-simplex $[v_0, v_1, v_2]$ in our space $X$, then we don't really have a hole in $X$, and on the other hand in homology our cycle will be a boundary, hence a zero class in $H_1(X)$. But if, on contrary, we cannot find the bigger simplex to fill this hole in $X$, this cycle won't be a boundary anymore, hence it will define a non-trivial element of $H_1(X)$.
Hence a non-trivial element of $H_n(X)$ indicates that we found something like an "$n$-th dimensional hole" in the space $X$. Because of that homology groups give an algebraic object which reflects topological properties of $X$.

Let us continue the study of general abstract homological algebra.
Suppose $(C, \partial)$ and $(C', \partial')$ are chain complexes. Suppose that for every $n \in \mathbb{Z}$ a group homomorphism $f_n \colon C_n \to C'_n$ is given. The collection $f = \{f_n \mid n \in \mathbb{Z}\}$ is called **a chain mapping** if its **components** $f_n$ commute with boundary operators i.e. if

$$f_{n-1} \circ \partial_n = \partial'_n \circ f_n.$$

This can also be illustrated by the commutativity of the diagram

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\partial"] & C'_n \arrow[d, "\partial'"] \\ C_{n-1} \arrow[r, "f_{n-1}"] & C'_{n-1}. \end{tikzcd}$$

Let $f \colon C \to C'$ be a chain mapping. Let $x \in Z_n(C)$. Then

$$\partial' f_n(x) = f_{n-1}\partial(x) = 0,$$

so $f_n(x) \in Z_n(C')$. Suppose $x = \partial_{n+1} y$ i.e. $x \in B_n(C)$. Then

$$f_n(x) = f_n \partial(y) = \partial'(f_{n+1} y) \in B_n(C').$$

In other words $f_n$ takes cycles to cycles and boundaries to boundaries, hence induces the group homomorphism $f_* \colon H_n(C) \to H_n(C')$.

Chain mappings can be composed - if $f \colon C \to C'$ and $g \colon C' \to C''$ are chain mappings, their **composite** is a chain mapping $g \circ f \colon C \to C''$, which components are naturally $g_n \circ f_n$ (check that this collection indeed satisfies the conditions for a chain mapping).
Also for every chain complex $C$ there is an **identity chain mapping** id defined degreewise to be the identity homomorphism $\mathrm{id}_n \colon C_n \to C_n$.
A chain mapping $f \colon C \to C'$ is called **an isomorphism of chain complexes** if there is a chain mapping $g \colon C' \to C$ (**an inverse** of $f$) such that $g \circ f = \mathrm{id} = f \circ g$. Clearly a chain mapping $f$ is an isomorphism if and only if it is a bijection in every degree. Its inverse is unique (because it is unique in every degree).

**Example 2.1.16.** *Suppose $f \colon X \to Y$ is a continuous mapping. If $\sigma \in \mathrm{Sing}_n(X)$ is a singular $n$-simplex $\sigma \colon \Delta_n \to X$, then the composite*

$$f_\sharp(\sigma) = f \circ \sigma \in \mathrm{Sing}_n(Y)$$

*is clearly a singular $n$-simplex in $Y$. Extend this mapping to a unique group homomorphism $f_\sharp = (f_\sharp)_n \colon C_n(X) \to C_n(Y)$. Then the collection $f_\sharp = ((f_\sharp)_n)$ is a chain mapping $C(X) \to C(Y)$ (Exercise 2.12a).*

*The homomorphism $(f_\sharp)_* \colon H_n(X) \to H_n(Y)$ will be denoted simply by $f_*$.*

Correspondence $f \mapsto f_*$ is functorial, in the following sence.

**Lemma 2.1.17.** *1)Suppose $f \colon X \to Y$ and $g \colon Y \to Z$ are continuous mappings. Then in the singular homology we have*

$$(g \circ f)_* = g_* \circ f_*.$$

*2) For identity mapping $\mathrm{id} \colon X \to X$ the induced mapping $\mathrm{id}_* \colon H_n(X) \to H_n(X)$ is the identity homomorphism $\mathrm{id} \colon H_n(X) \to H_n(X)$.*

*Proof.* Exercise 2.12b). $\square$

This implies that singular homology is a topological invariant.

**Corollary 2.1.18.** *Suppose $f \colon X \to Y$ is a homeomorphism. Then $f_* \colon H_n(X) \to H_n(Y)$ is an isomorphism for all $n \in \mathbb{N}$.*

*Proof.* Let $g\colon Y \to X$ be the inverse of $f$. Then

$$\mathrm{id} = \mathrm{id}_* = (g \circ f)_* = g_* \circ f_*,$$

and similarly $f_* \circ g_* = \mathrm{id}$.Hence $g_*$ is the inverse of $f_*$.

□

Thus homeomorphic spaces have isomorphic singular homology groups. The same is not so obvious for the simplicial homology. Indeed suppose $K$ and $K'$ are $\Delta$-complexes that have the same polyhedra, i.e. $|K|$ and $|K'|$ are homeomorphic as spaces. Are $H_n(K)$ isomorphic to $H_n(K')$ for all $n \in \mathbb{N}$? At this point the answer is not at all obvious. There are different ways to triangulate a space, why would corresponding homologies have anything in common? On the other hand if simplicial homology would not be a topological invariant, it wouldn't be of any good for topologist and would not help us study topological spaces.

Later we will prove that it is indeed a topological invariant and in fact the following is true.

**Proposition 2.1.19.** *Suppose $K$ is a $\Delta$-complex. The inclusion $i\colon C(K) \hookrightarrow C(|K|)$ (which is a chain mapping) induces isomorphisms in homology i.e. $i_*\colon H_n(K) \to H_n(|K|)$ is an isomorphism for all $n \in \mathbb{N}$*

We will prove the proposition 2.1.19 (at least for finite complexes, which is enough for our purposes and applications) later, after we have developed enough machinery. Nevertheless we will from now on keep in mind this result, which allows us to compute (singular) homology groups of some spaces by triangulating them. Examples will be studied at the end of this section.

**Subcomplexes and quotient complexes**.

Let $(C, \partial)$ be a chain complex. Supose for every $n \in \mathbb{Z}$ we are given a subgroup $C'_n$ of $C_n$ such that $\partial_n(C'_n) \subset C'_{n-1}$ for all $n \in \mathbb{Z}$. Then the collection $C' = \{C'_n\}_{n\in\mathbb{Z}}$ together with the restriction $\partial'_n = \partial_n|C'_n$ as the boundary operator for every $n \in \mathbb{N}$ obviously defines a chain complex $(C', \partial')$. We say that $(C', \partial)$ is **a chain subcomplex** of the chain complex $(C, \partial)$. The collection $i = \{i_n\colon C'_n \to C_n\}$ of inclusions obviously defines a chain mapping $i\colon C' \to C$. We also say that $(C, C')$ is a pair of chain complexes.

Suppose $(C', \partial')$ is a subcomplex of $(C, \partial)$. Since $\partial(C'_n) \subset C'_{n-1}$ for all $n \in \mathbb{N}$, homomorphism $\partial$ induces a homomorphism

$$\overline{\partial}_n\colon C_n/C'_n \to C_{n-1}/C'_{n-1}$$

for all $n \in \mathbb{N}$. Hence if we denote $\overline{C}_n = C_n/C'_n$, we obtain a chain complex $(\overline{C}, \overline{\partial})$, which is called **a quotient chain complex** of $(C, \partial)$ and can also be denoted by $C/C'$.
The quotient mappings $p_n\colon C_n \to \overline{C}_n$ define a chain mapping $p\colon C \to \overline{C}$.

**Examples 2.1.20.** *1) Suppose $L$ is a $\Delta$-subcomplex of a $\Delta$-complex $K$. Since the set of generators of $C_n(L)$ is a subset of the set of generators of $C_n(K)$ we can consider $C_n(L)$ a subgroup of $C_n(K)$ for every $n \in \mathbb{Z}$. Since the boundary operator on $C_n(L)$ is obviously the same as the restriction of the boundary operator on $C_n(K)$, we see that $C(L)$ is a subcomplex of $C(K)$. The corresponding quotient complex $C(K)/C(L)$ is denoted $C(K, L)$. It is easy to see that this is a free complex with the set of free generators in one-to-one correspondence with the set of the geometric $n$-simplices of $K \setminus L$.*
*The $n$-th homology group of $C(K, L)$ is denoted $H_n(K, L)$ and is called* **the $n$-th relative homology group of the pair** *$(K, L)$.*

*2) Suppose $A$ is a subspace of a a topological space $X$. An element of $\mathrm{Sing}_n(A)$ is a continuous mapping $f\colon \Delta_n \to A$ which can be identified with an element $f \in \mathrm{Sing}_n(X)$, with $f(\Delta_n) \subset A$. Conversely any such element definies a unique element of $\mathrm{Sing}_n(A)$ in an obvious way. Hence the standard set of the free generators of $C_n(A)$ can be identified with the subset of the standard set of the generators of $C_n(X)$ and thus we can consider $C_n(A)$ as a subgroup of $C_n(X)$ for all $n \in \mathbb{Z}$.*
*Corresponding quotient complex $C(X)/C(A)$ is denoted $C(X, A)$. It is easy to see that it is a free complex, with the set of generators of $C_n(X, A)$ being the set of all continuous mappings $f\colon \Delta_n \to X$ with the property*

$$\mathrm{Im}\, f \cap X \setminus A \neq \emptyset.$$

*The $n$-th homology group of $C(X, A)$ is denoted $H_n(X, A)$ and is called* **the relative $n$-th homology group of the pair** *$(X, A)$.*

*The groups $H_n(K)$ and $H_n(X)$ defined earlier are often referred to as the "absolute" homology groups, as opposed to "relative " groups defined for pairs. However notice that absolute groups can be considered a special case of relative groups - just take $L$ or $A$ to be empty subcomplex/empty subspace, then corresponding complexes $C(L)$, $C(A)$ are zero complexes, so $C(K, L)$ and $C(X, A)$ can be identified with $C(K)$ and $C(X)$. It follows that $H_n(K) = H_n(K, \emptyset)$ and $H_n(X) = H_n(X, \emptyset)$.*
*Hence from the technical point of view it is enough to consider the relative*

*homology groups.*

*3) As we have already seen, $C(K)$ is a subcomplex of $C(|K|)$ for every $\Delta$-complex $K$.*

Suppose $(C, C')$ and $(D, D')$ are pairs of chain complexes and $f\colon C \to D$ is a chain mapping, such that $f$ maps $C'$ into $D'$, hence defines by restriction a chain mapping $f|\colon C' \to D'$. In this case we say that $f$ is **a chain mapping of pairs** and denote $f\colon (C, C') \to (D, D')$.
Since $f(C') \subset D'$, $f$ defines a chain mapping $\overline{f}\colon C/C' \to D/D'$ in quotient chain complexes.

**Example 2.1.21.** *Suppose $(X, A)$, $(Y, B)$ are topological pairs. The mapping $f\colon X \to Y$ which maps $A$ into $B$ is called* **a mapping of topological pairs**. *This is also denoted as $f\colon (X, A) \to (Y, A)$. In this case $f$ defines by restriction the mapping $f|\colon A \to B$.*

*The induced chain mapping $f_\sharp\colon C(X) \to C(Y)$ obviously maps $C(A)$ into $C(B)$. Hence we have the induced homomorphism $f_\sharp\colon C(X, A) \to C(Y, B)$. This mapping, in its turn, induces homomorphisms in relative homology. These homomorphisms will be also denoted as $f_*\colon H_n(X, A) \to H_n(Y, B)$.*

To prevent us from completely sliding into theoretical abstract nonsence, we will now step away from our general course to make some concrete computations of homology groups, using $\Delta$-complexes. This will also provide some real feel and taste for the algebra we attempt to use.

**Example 2.1.22.** *Let us start off with the 2-simplex $\Delta = [a, b, c]$ and its boundary $\partial\Delta$ considered as $\Delta$-complexes (no identifications). For $n < 0$ or $n > 2$ both complexes have no simplices in dimension $n$, so $C_n(\Delta) = C_n(\partial\Delta) = 0$ for these values of $n$. For $n = 0$ or $n = 1$ both complexes have the same set of simplices, $C_0$ is a free group on 3 free generators $a, b, c$ and $C_1$ is also a free group on 3 generators $[a, b]$, $[b, c]$ and $[a, c]$. In dimension 2 $\Delta$ has one simplex, so $C_2(\Delta)$ is a free group generated by a single element $[a, b, c]$, while $C_2(\partial\Delta) = 0$.*
*Let's start with $H_2$. For $\partial\Delta$ it is obviously zero, since $C_2 = 0$. For $\Delta$ we have*

$$\partial_2([a, b, c]) = [a, b] + [b, c] - [a, c] = x,$$

*where $x \in C_1$ is obviously a non-zero element. Since in free group $x \neq 0$ implies $nx \neq 0$ for $n \neq 0$, we see that $\partial_2$ is injective, hence $Ker\partial_2 = 0$. It follows that $H_2(\Delta)$ is also zero, as a quotient of a zero group.*

*Next let us investigate $H_1$. Boundary operator $\partial_1$ is the same in both case, defined by*

$$\partial_1(n[a,b]+m[b,c]+l[a,c]) = n(b-a)+m(c-b)+l(c-a) = (n-m)b+(m+l)c-(n+l)a.$$

*Hence for $x = n[a,b] + m[b,c] + l[a,c])$ the condition $x \in \operatorname{Ker}\partial_1$ is equivalent to $n - m = m + l = n + l = 0$, which is the same as $n = m = -l$. It follows that $\operatorname{Ker}\partial_1$ is a free group on $1$ element, generated by $[a + b] + [b, c] - [a, c]$. Since in $\Delta$ this element is precisely the boundary of the only generator $[a, b, c]$, we see that $H_1(\Delta) = 0$. On the other hand in $\partial\Delta$ there are no $2$-simplices, so $\operatorname{Im}\partial_2 = 0$, hence $H_1(\partial\Delta) = \operatorname{Ker}\partial_1$ is a free group on $1$ generator $[a+b]+[b,c]-[a,c]$. This illustrates precisely the idea of homology - we have detected a $1$-dimensional hole in the boundary of triangle, which is represented by the closed loop, that goes around it. In the triangle this hole is stuffed with the interior of triangle, so the hole itself vanishes, and the corresponding homology group is trivial.*

*It remains to calculate $H_0$. Since both complexes have the same $1$-skeleton, this will be the same for both. $\partial_0$ is of course zero mapping (since $C_{-1} = 0$), so $\operatorname{Ker}\partial_0 = C_0$ is a free group on $3$ elements $a, b, c$. The image of $\partial_1$ consists of the subgroup generated by the elements $\partial[a, b] = b - a$, $\partial[a, c] = c - a$ and $\partial[b, c] = b - c$. Since*

$$c - b = (c - a) - (b - a),$$

*this is the same as the subgroup generated by $c - a$ and $b - a$. One can show that $\{c - a, b - a, a\}$ is a basis of the free group $C_0$ (Exercise 2.14a). Hence it follows that*

$$H_0 = C_0/\operatorname{Im}\partial_1 = (\mathbb{Z}(a) \oplus \mathbb{Z}(c-a) \oplus \mathbb{Z}(b-a))/(\mathbb{Z}(c-a) \oplus \mathbb{Z}(b-a)) \cong \mathbb{Z}(a)$$

*is a free group on $1$ generator.*

*Since the boundary of a triangle is isomorphic to $S^1$ we have calculated the homology groups of the circle, assuming we believe our simplicial homology is topological invariant.*
*Another obvious way to triangulate $S^1$ is to take one $1$-simplex and identify its ends. As an exercise calculate the simplicial homology of this representation and verify that it gives the same groups as above.*

*It would be difficult to calculate homology of $\Delta_n$ and $\partial\Delta_n$ in the same brute-de-force way, but it is easy to calculate the relative homology of the pair $(\Delta_n, \partial\Delta_n)$. Indeed both have the same simplices in all dimensions except $n$,* *hence $C_k(\Delta_n, \partial\Delta_n) = 0$ for $k \neq n$, while $C_n(\Delta_n, \partial\Delta_n)$ is clearly isomorphic to $\mathbb{Z}$, generated by the $n$-simplex $\Delta_n$ itself. It follows easily that the homology of this complex has the same desciption. In other words*

$$H_m(\Delta_n, \partial\Delta_n) \cong \begin{cases} \mathbb{Z}, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

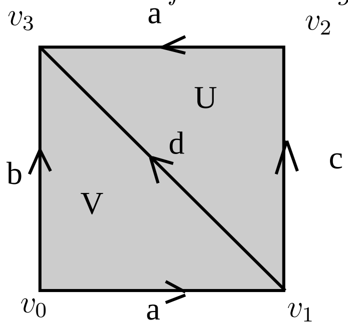
*In the same fashion one can compute homology of the pair $(K^n, K^{n-1})$ for any $\Delta$-complex $K$ (exercise 2.16), to obtain*

$$H_m(K^n, K^{n-1}) = \begin{cases} \mathbb{Z}^{(A)}, & \text{if } m = n, \\ 0, & \text{otherwise,} \end{cases}$$

*where $A$ is the set of (geometric) $n$-simplices of $K$.*

*Later we will use this simple result in the proof of the theorem 2.1.19.*

**Example 2.1.23. Mobius band**.
*Let us calculate the homology of the Mobius band, using the familiar $\Delta$-complex structure, obtained from a square, as in the picture below. Notice that the order of vertices is given by their integer indices.*




*Now there are 2 triangles $U$ and $V$, 4 edges $a, b, c, d$ and 2 vertices - $v_0 = v_2 = x$ and $v_1 = v_3 = y$. First we calcluate $H_2 = \operatorname{Ker} \partial_2$. Now*

$$\partial(nU+mV) = n(a-d+c)+m(d-b+a) = (n+m)a+(m-n)d+nc-mb = 0$$

*if and only if $n = m = n+m = n-m = 0$, hence $n = m = 0$. In other words $\operatorname{Ker} \partial_2$ is trivial, so $H_2 = 0$. What about the image of $\partial_2$? By the calulation above it consists of the points*

$$\{nc+mb+(n-m)a+(n+m)d = n(c+a+d)+m(b-a+d) | n, m \in \mathbb{Z}\} \subset C_1$$

*(the sign of $m$ is switched for the convinience), so it clearly is a subgroup generated by $a + c + d$ and $b - a + d$ (just put $n = 1$, $m = 0$ and $n = 0$,*

*$m = 1$).*
*On the other hand*

$$\partial_1(na+mb+kc+ld) = n(y-x)+m(y-x)+k(x-y)+l(y-y) = (n+m-k)y+(k-n-m)x = 0$$

*if and only if $n+m-k = 0 = k-n-m = -(n+m-k)$, hence if and only if $k = n+m$. Thus*

$$\operatorname{Ker}\partial_1 = \{na+mb+(n+m)c+ld = n(a+c)+m(b+c)+ld | n,m,l \in \mathbb{Z}\}$$

*is a free group on 3 generators $a+c, b+c, d$ (as an exercise you can check that these elements are independent, also we don't really need that information). Now $a+c = (a+c+d)-d$ and $b+c = (b-a+d)+(a+c+d)-2d$, so the group generated by $a+c, b+c$ and $d$ is contained in the free group generated by $a+c+d, b-a+d$ and $d$ (as another exercise check that these elements are independent!). Conversely $a+c+d = (a+c)+d$ and $b-a+d = (b+c)-(a+c)+d$, so the group generated by $a+c+d, b-a+d, d$ is contained in the group generated by $a+c, b+c$ and $d$. Thus*

$$\operatorname{Ker}\partial_1 = \mathbb{Z}[a+c+d] \oplus \mathbb{Z}[b-a+d] \oplus \mathbb{Z}[d].$$

*Hence*

$$H_1 = \operatorname{Ker}\partial_1/\operatorname{Im}\partial_2 = (\mathbb{Z}[a+c+d]\oplus\mathbb{Z}[b-a+d]\oplus\mathbb{Z}[d])/(\mathbb{Z}[a+c+d]\oplus\mathbb{Z}[b-a+d]) \cong \mathbb{Z}[d] \cong \mathbb{Z}.$$

*Notice that the homology group $H_1$ of the Mobius band is generated by the class of the idge $d$ i.e. the diagonal of the square (which looks like the circle, since its end points are identified).*
*It remains to calculate $H_0 = (\mathbb{Z}[x] \oplus \mathbb{Z}[y])/\operatorname{Im}\partial_1$. Since*

$$\partial_1(na+mb+kc+ld) = n(y-x)+m(y-x)+k(x-y)+l(y-y) = (n+m-k)y+(k-n-m)x =$$

$$= lx - ly = l(x-y),$$

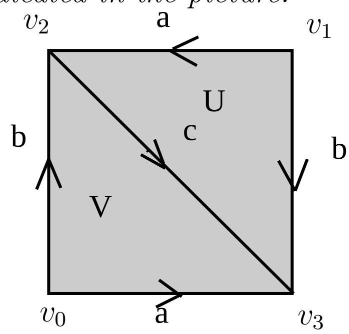
*where $l = n+m-k$, it follows that $\operatorname{Im}\partial_1 = \mathbb{Z}[x-y]$. It is easy to check that $\{x-y, y\}$ is also a basis for $C_0$. Hence it follows that*

$$H_0 = (\mathbb{Z}[x-y] \oplus \mathbb{Z}[y])/\mathbb{Z}[x-y] \cong \mathbb{Z}[y] \cong \mathbb{Z}.$$

*Both classes $[x]$ and $[y]$ generate $H_0$. For $[x]$ it follows from the equation $[x] = [x-y]+[y] = [y]$, where the fact that $x-y$ is a boundary in $C_0$ is used.*

*Observe that $S^1$ and the Mobius band have the same homology groups. This is not a coincidence - although they are not homeomorphic as spaces, they have* **the same homotopy type** *(proof of both claims will be an exercise given to you later), and we will prove in the next part of the course that the spaces with the same homotopy type has the same homology groups.*

**Example 2.1.24.** *So far all the homology groups we have calculated were simple free groups. As a more sophisticated example let us calculate the homology of the projective plane using the familiar $\Delta$-complex structure as indicated in the picture.*




*Now* $C_2 = \mathbb{Z}[U] \oplus \mathbb{Z}[V]$ *and*

$$\partial_2(nU+mV) = n(c-b+a)+m(c-a+b) = (n-m)a+(m-n)b+(n+m)c = 0$$

*if and only if* $n+m = n-m = 0$ *i.e. if* $n = m = 0$. *Hence* $\partial_2$ *is injective, so its kernel, and consequently* $H_2$*, are zero.*
*Also* $\operatorname{Im}\partial_2$ *is a subgroup generated by* $c-b+a = u$ *and* $c-a+b = v$.
*Denote* $v_0 = v_1 = x$, $v_2 = v_3 = y$ *and observe that*

$$\partial_1(na+mb+lc) = n(y-x)+m(y-x) = (n+m)(y-x) = 0$$

*if and only if* $n = -m$. *hence*

$$\operatorname{Ker}\partial_1 = \{n(a-b)+lc \mid n, l \in \mathbb{Z}\},$$

*so that* $\operatorname{Ker}\partial_1$ *is a free group generated by* $a-b$ *and* $c$.
*Now we use the fact (exercise 2.17) that if* $\{\alpha, \beta\}$ *is a basis of a free abelian group, also* $\{\alpha \pm \beta, \beta\}$ *is a basis. We apply it first to the elements* $\{c-b+a, c-a+b\}$ *(check that they are independent, hence are a basis of the group* $\operatorname{Im}\partial_2$ *they generate!) to obtain the basis* $\{2c, c-(a-b)\}$ *for* $\operatorname{Im}\partial_2$. *Also the same fact applayed to the basis* $\{a-b, c\}$ *(check that it is a basis!) of* $\operatorname{Ker}\partial_1$ *gives the basis* $\{c, c-(a-b)\}$ *for* $\operatorname{Ker}\partial_1$. *Hence*

$$H_1 = (\mathbb{Z}[c]\oplus\mathbb{Z}[c-(a-b)])/(\mathbb{Z}[2c]\oplus\mathbb{Z}[c-(a-b)]) \cong \mathbb{Z}[c]/\mathbb{Z}[2c] \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.$$

*Thus the first homology group of the projective plane is a group of two elements, generated by the only non-trivial element* $[c]$. *Since* $[c] = [c-(a-b)]+[a-b]$*, and* $c-(a-b)$ *is a boundary element, it follows that* $[c] = [a-b]$*, so we can think of our generator as the image of the " half-ark " on the boundary of the disk* $overline B^2$. *The homology class of this ark is not trivial, but if* *we add it to itself, thus "travelling" it twice - back and forth as the picture indicates - we obtain a trivial path.*

*It remains to calculate the* 0*-th homology. Again* $H_0 = (\mathbb{Z}[x] \oplus \mathbb{Z}[y]) / \operatorname{Im} \partial_1$*. Since*

$$\partial_1(na + mb + lc) = (n + m)(y - x),$$

*we see that* $\operatorname{Im} \partial_1 = \mathbb{Z}[y - x]$*. Since* $\{y - x, x\}$ *is a basis for* $C_0$*, we see as above that* $H_0 \cong \mathbb{Z}[x] = \mathbb{Z}[y] \cong \mathbb{Z}$*.*

Further examples are given to you as exercises 2.19, 2.20, 2.21.

## 2.2 Some (homological) algebra

Suppose we have a sequence

$$\ldots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \ldots$$

of abelian groups and homomorphisms. It can be unlimited in both direction (i.e. indexed on $\mathbb{Z}$) or stop somewhere on the left or/and on the right. We say that this sequence is **exact** at $A_n$ if

$$\operatorname{Ker} f_n = \operatorname{Im} f_{n+1}.$$

If the sequence is exact at its everygroup, we say that this sequence is **an exact sequence** (of abelian groups).

Since the condition $\operatorname{Im} f_{n+1} \subset \operatorname{Ker} f_n$ is equivalent to the condition $f_{n+1} \circ f_n = 0$, we see that the sequence above is exact at $A_n$ if and only
1)$f_{n+1} \circ f_n = 0$ and
2)$\operatorname{Ker} f_n \subset \operatorname{Im} f_{n+1}$.

Let $(C, \partial)$ be a chain complex. We can think of it as an unlimited sequence

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots$$

of groups and homomorphisms. Now the condition $\partial_{n+1} \circ \partial_n = 0$ is equivalent to the condition

$$\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n.$$

It follows that this sequence is exact at $C_n$ if and only $H_n(C) = 0$. Hence in some sence homology groups of a chain complex measure the extend to which the complex, thought of as a sequence as above, fails to be exact.

A chain complex $(C, \partial)$ is called **acyclic** if it is exact as a sequence. From the previous considerations we see that $(C, \partial)$ is acyclic if and only if $H_n(C) = 0$ for all $n \in \mathbb{N}$.

An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

is called a **short exact sequence** (of abelian groups).

**Example 2.2.1.** *Suppose $A$ is a subgroup of $B$. Denote the inclusion mapping by $i\colon A \to B$ and let $p\colon B \to B/A$ be the canonical projection to the quotient group. Then the sequence*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} B/A \longrightarrow 0$$

*is exact.*

**Lemma 2.2.2.** *A sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*is short exact if and only if*
*1) $f$ is injection,*
*2) $g$ is surjection,*
*3)* $\operatorname{Im} f = \operatorname{Ker} f$

*Proof.* Exactness at $A$ means that $\operatorname{Im}(0) = 0 = \operatorname{Ker} f$, which means precisely that $f$ is injection. Likewise exactness at $C$ means that $\operatorname{Im} g = \operatorname{Ker} 0 = C$, i.e. $g$ is surjective. $\square$

It follows that every short exact sequence is essentially of the form 2.2.1. Indeed since $f$ is an injection, we can identify $A$ with a subrgroup $\operatorname{Im} f$ of $B$, so $f$ becomes an inclusion under this identification. Since $g$ is surjective and its kernel equals a subgroup $A$, the first isomorphism theorem of the group theory says that $g$ defines an isomorpism $A/B \cong C$. Under this identification $g$ corresponds to the canonical projection $p$.

A sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

of chain complexes and chain mappings is called **a short exact sequence** if it is exact in every dimension as the sequence of abelian groups and homomorphisms i.e. if the sequence

$$0 \longrightarrow C'_n \xrightarrow{f} C_n \xrightarrow{g} \overline{C}_n \longrightarrow 0$$

is short exact for every $n \in \mathbb{Z}$.
It follows that in this case $C'$ can be considered a subcomplex of $C$ and $\overline{C}$ can be identified with a quotient subcomplex $C/C'$. In other words the sequence is essentially isomorphic to the sequence

$$0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} \overline{C} \longrightarrow 0$$

where $i \colon C' \hookrightarrow C$ is an inclusion of s subcomplex and $p \colon C \to C/C'$ is a canonical projection.

Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes. We shall construct for every $n \in \mathbb{Z}$ a canonical mapping

$$\partial \colon H_n(\overline{C}) \to H_{n-1}(C'),$$

called **the boundary operator** induced by this sequence.

Suppose $x \in \overline{C}_n$ is a cycle, i.e. an element of $\mathrm{Ker}(\overline{\partial}_n)$. Since $g_n$ is a surjection, there exists an element $y \in C_n$ such that $g_n(y) = x$. Then

$$g_n(\partial_n(y)) = \overline{\partial}_n(g_n(y)) = \overline{\partial}_n(x) = 0.$$

Since the sequence is exact this means that there is an element $z \in C'_{n-1}$ such that $f_{n-1}(z) = \partial_n(y)$. Moreover $z$ is unique, since $f_{n-1}$ is an injection. Let us show that $z$ is a cycle. We have

$$f_{n-2}\partial'_{n-1}(z) = \partial_{n-1}(f_{n-1}(z)) = \partial_{n-1}\partial_n(y) = 0.$$

Since $f_{n-2}$ is an injection, it follows that $\partial'_{n-1}(z) = 0$ i.e. $z$ in indeed a cycle in $C'_{n-1}$. Hence the class $z \in H_{n-1}(C')$ is defined. We assert

$$\Delta(x) = [z].$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C'_{n+1} & \xrightarrow{f} & C_{n+1} & \xrightarrow{g} & \overline{C}_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial'_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \overline{\partial}_{n+1}} & & \\
0 & \longrightarrow & C'_{n} & \xrightarrow{f} & C_{n} & \xrightarrow{g} & \overline{C}_{n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial'_{n}} & & \downarrow{\scriptstyle \partial_{n}} & & \downarrow{\scriptstyle \overline{\partial}_{n}} & & \\
0 & \longrightarrow & C'_{n-1} & \xrightarrow{f} & C_{n-1} & \xrightarrow{g} & \overline{C}_{n-1} & \longrightarrow & 0
\end{array}$$

Naturally we want $\Delta\colon Z_n(\overline{C}) \to H_{n-1}(C')$ to be a well-defined mapping. Our construction involved a choice of $y \in C_n$, so we need to show that $\Delta(x)$ does not depend on this choice. Suppose $y' \in C_n$ is another element such that $g_n(y') = x$ and let $z' \in Z'_{n-1}$ be the unique element with $f_{n-1}(z') = \partial_n(y')$. Since $g_n(y) = g_n(y')$, it follows that $y - y' \in \operatorname{Ker} g_n = \operatorname{Im} f_n$, so there is $u \in C'_n$ such that $f_n(u) = y - y'$. Now

$$f_{n-1}\partial'_n(u) = \partial_n(f_n(u)) = \partial_n(y) - \partial_n(y') = f_{n-1}(z - z').$$

Since$f_{n-1}$ is an injection, it follows that

$$\partial'_n(u) = z - z',$$

hence $z - z' \in B_{n-1}(C')$, so $[z] = [z']$ in $H_{n-1}(C')$. We have proved that the construction as above defines a mapping $\Delta\colon Z_n(\overline{C}) \to H_{n-1}(C')$.

**Lemma 2.2.3.** *The mapping $\Delta$ is a homomorphism and factors through $B_n(\overline{C})$, hence defines a homomorphism*

$$\partial\colon H_n(\overline{C}) \to H_{n-1}(C').$$

*Proof.* Suppose $x, x' \in Z_n(\overline{B})$. Let $y, y' \in C_n$ such that $g(y) = x$, $g(y') = x'$. Let $z, z' \in C'_{n-1}$ be such that $f(z) = \partial_n(y), f(z') = \partial_n(y')$. Then $f(z + z') = \partial_n(y + y')$ and $g(x + x') = y + y'$. Thus

$$\Delta(x + x') = [z + z'] = [z] + [z'] = \Delta(x) + \Delta(x').$$

Hence $\Delta$ is a group homomorphism. Suppose $x \in B_n(\overline{C})$ and let $w \in \overline{C}_{n+1}$ be such that $\overline{\partial}_{n+1}(w) = x$ and $v \in C_{n+1}$ be such that $g(v) = w$. Then

$$g(\partial_{n+1}v) = \overline{\partial}_{n+1}(g(v)) = \overline{\partial}_{n+1}(w) = x,$$

hence we can choose $y = \partial_{n+1}v$ to be the element of $C_n$ with $g(y) = x$. Now $\partial_n(y) = \partial_n\partial_{n+1}v = 0$, so $\Delta(x) = 0$, by the definition. $\square$

The homomorphism $\partial$ is **natural** in the following sense.

**Lemma 2.2.4.** *Suppose*

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, "f"] \arrow[d, "\alpha"] & C \arrow[r, "g"] \arrow[d, "\beta"] & \overline{C} \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & D' \arrow[r, "f'"] & D \arrow[r, "g'"] & \overline{D} \arrow[r] & 0 \end{tikzcd}$$

*is a commutative diagram of chain complexes and chain mappings with exact rows.*
*Then the diagram*

$$\begin{tikzcd} H_n(\overline{C}) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_{n-1}(C') \arrow[d, "\alpha_*"] \\ H_n(\overline{D}) \arrow[r, "\partial"] & H_{n-1}(D') \end{tikzcd}$$

*is commutative.*

*Proof.* Exercise 2.22a). □

One of the most important basic results in homological algebra is the existence of **the long exact sequence of homology groups**, induced by the short exact sequence of chain complexes.

**Theorem 2.2.5.** *Suppose*

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

*is a short exact sequence of chain complexes. Then the sequence*

$$\ldots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\partial} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\partial} H_{n-1}(C') \longrightarrow \ldots$$

*is an exact sequence of abelian groups and homomorphisms.*

*Proof.* 1) Exactness at $H_n(C)$:
Since $g \circ f = 0$, in homology we have $g_* \circ f_* = (g \circ f)_* = 0$. Conversely suppose $x \in Z_n(C)$ is such that $g_*[x] = 0 \in H_n(\overline{C})$. This means that $g(x) = \overline{\partial}_{n+1} w$ for some $w \in \overline{C}_{n+1}$. Let $v \in C_{n+1}$ be such that $g(v) = w$. Then

$$g(\partial_{n+1} v) = \overline{\partial}_{n+1} g(v) = g(x),$$

hence $x - \partial_{n+1} v \in \operatorname{Ker} g = \operatorname{Im} f$. Consequently there is $z \in C'_n$ such that

$$x - \partial_{n+1} v = f(z).$$

Since $f$ is an injection it follows easily that $z$ is a cycle; indeed

$$f(\partial'_n(z)) = \partial_n f(z) = \partial_n(x - \partial_{n+1} v) = 0.$$

Thus there is $[z] \in H_n(C')$ and

$$f_*[z] = [f(z)] = [x - \partial_{n+1} v] = [x],$$

since boundary element $\partial_{n+1} v$ becomes zero in homology.
We have proved that $\operatorname{Ker} g_* \subset \operatorname{Im} f_*$.

2)Exactness at $H_n(\overline{C})$: first we prove that $\Delta \circ g_* = 0$. This is straightforward - suppose $y \in Z_n(C)$, and let $x = g(y)$. Then $\Delta(g_*([y])) = \Delta([x]) = 0$, since $\partial_n(y) = 0$.
The proof of the inclusion

$$\operatorname{Ker} \Delta \subset \operatorname{Im} g_*$$

is left as an exercise.

3) Exactness at $H_n(C')$: Exercise. □

Long exact sequence is natural.

**Proposition 2.2.6.** *Suppose*

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r, "f"] \arrow[d, "\alpha"] & C \arrow[r, "g"] \arrow[d, "\beta"] & \overline{C} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D' \arrow[r, "f'"] & D \arrow[r, "g'"] & \overline{D} \arrow[r] & 0
\end{tikzcd}$$

*is a commutative diagram of chain complexes and chain mappings with exact rows.*
*Then the diagram*

$$\begin{tikzcd}
\ldots \arrow[r] & H_{n+1}(\overline{C}) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_n(C') \arrow[r, "f_*"] \arrow[d, "\alpha"] & H_n(C) \arrow[r, "g_*"] \arrow[d, "\beta_*"] & H_n(\overline{C}) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_{n-1}(C') \arrow[r] \arrow[d, "\alpha"] & \ldots \\
\ldots \arrow[r] & H_{n+1}(\overline{D}) \arrow[r, "\partial"] & H_n(D') \arrow[r, "f_*"] & H_n(D) \arrow[r, "g_*"] & H_n(\overline{D}) \arrow[r, "\partial"] & H_{n-1}(D') \arrow[r] & \ldots
\end{tikzcd}$$

*is commutative.*

*Proof.* Exercise 2.23b). $\square$

**Examples 2.2.7.** *Let $(X, A)$ be a topological pair. Then by definitions the exact sequence*

$$0 \longrightarrow C(A) \xrightarrow{i_\sharp} C(X) \xrightarrow{j_\sharp} C(X, A) \longrightarrow 0$$

*is a short exact sequence of chain complexes. Here $i\colon A \hookrightarrow X$ is an inclusion and $j\colon (X, \emptyset) \to (X, A)$ is a map of pairs.*
*By the theorem 2.2.5 there is a long exact sequence*

$$\ldots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \ldots$$

*of singular homology groups. This exact sequence will be referred to as* **the long exact homology sequence of the pair** *$(X, A)$.*

*Suppose $f\colon (X, A) \to (Y, B)$ is a continuous mapping of the topological pairs. Then $f$ induces chain mappings $f_\sharp$ between chain complexes $C(X) \to C(Y)$, $C(A) \to C(B)$ and $C(X, A) \to C(Y, B)$, and the diagram*

$$\begin{tikzcd}
0 \arrow[r] & C(A) \arrow[r, "i_\sharp"] \arrow[d, "f|_\sharp"] & C(X) \arrow[r, "j_\sharp"] \arrow[d, "f_\sharp"] & C(X, A) \arrow[r] \arrow[d, "f_\sharp"] & 0 \\
0 \arrow[r] & C(B) \arrow[r, "i_\sharp"] & C(Y) \arrow[r, "j_\sharp"] & C(Y, B) \arrow[r] & 0
\end{tikzcd}$$

*commutes. Hence Proposition 2.2.6 implies that there is a commutative diagram*

$$\begin{tikzcd}
\ldots \arrow[r] & H_{n+1}(X, A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f|_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X, A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f|_*"] & \ldots \\
\ldots \arrow[r] & H_{n+1}(Y, B) \arrow[r, "\partial"] & H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "j_*"] & H_n(Y, B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \ldots
\end{tikzcd}$$

*which rows are the long exact homology sequences of the pairs $(X, A)$ and $(Y, B)$.*

*There is also a useful generalization for the triples.* **A topological triple** *is a triple $(X, A, B)$ of topological spaces where $B \subset A \subset X$. In this situation we have a short exact sequence (exercise)*

$$0 \longrightarrow C(A, B) \xrightarrow{i_\sharp} C(X, B) \xrightarrow{j_\sharp} C(X, A) \longrightarrow 0,$$

*where $i\colon (A, B) \to (X, B)$ and $j\colon (X, B) \to (X, A)$ are obvious inclusions. This implies the following result.*

**Lemma 2.2.8.** *Suppose $(X, A, B)$ is a topological triple. Then the sequence*

$$\dots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial'} H_n(A, B) \xrightarrow{i_*} H_n(X, B) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \xrightarrow{i_*} \dots$$

*is exact. This sequence is called* **the long exact homology sequence** *of the triple $(X, A, B)$. It is natural with respect to the mappings of triples. Moreover for the boundary operators of the long exact homology sequences of the pair $(X, A)$ and of the triple $(X, A, B)$ we have the commutative diagram*

$$\begin{tikzcd} & H_n(A) \arrow[dd, "i_*"] \\ H_{n+1}(X, A) \arrow[ur, "\partial"] \arrow[dr, "\partial'"'] & \\ & H_n(A, B), \end{tikzcd}$$

*where $i\colon A \to (A, B)$ is an inclusion.*

*Proof.* Exercise 2.25. $\square$

*Similarly for the pair of $\Delta$-complexes $(K, L)$ there is* **the long exact homology sequence of the pair** $(K, L)$

$$\dots \longrightarrow H_{n+1}(K, L) \xrightarrow{\partial} H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{j_*} H_n(K, L) \xrightarrow{\partial} H_{n-1}(L) \longrightarrow \dots$$

*For the canonical inlusions $k\colon C(K) \to C(|K|)$ of the groups of the simplicial chains into the groups of singular chains we have an obvious commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & C(L) \arrow[r, "i_\sharp"] \arrow[d, "k"] & C(K) \arrow[r, "j_\sharp"] \arrow[d, "k"] & C(K, L) \arrow[r] \arrow[d, "k"] & 0 \\ 0 \arrow[r] & C(|L|) \arrow[r, "i_\sharp"] & C(|K|) \arrow[r, "j_\sharp"] & C(|K|, |L|) \arrow[r] & 0 \end{tikzcd}$$

*which induces then commutative diagram*

$$\begin{tikzcd} \dots \arrow[r] & H_{n+1}(K, L) \arrow[r, "\partial"] \arrow[d, "k_*"] & H_n(L) \arrow[r, "i_*"] \arrow[d, "k_*"] & H_n(K) \arrow[r, "j_*"] \arrow[d, "k_*"] & H_n(K, L) \arrow[r, "\partial"] \arrow[d, "k_*"] & H_{n-1}(L) \arrow[r] \arrow[d, "k_*"] & \dots \\ \dots \arrow[r] & H_{n+1}(|K|, |L|) \arrow[r, "\partial"] & H_n(|L|) \arrow[r, "i_*"] & H_n(|K|) \arrow[r, "j_*"] & H_n(|K|, |L|) \arrow[r, "\partial"] & H_{n-1}(|L|) \arrow[r] & \dots \end{tikzcd}$$

*between long exact homology sequences.*

The last example also provides us with motivation for the next extremely useful algebraic result. As is already mentioned, eventially we want to prove that $k_*\colon H_n(K) \to H_n(|K|)$ is an isomorphism for every $n \in \mathbb{N}$. Now take a look at the last commutative diagram between long exact sequences of the pairs $(K, L)$ and $(|K|, |L|)$. Suppose we already know that the result is true for the subcomplex $L$ (for example in finite case $L$ could have less simplices than $K$, so we could use an inductive assumption) and for the pair $(K, L)$. Then in the diagram above all five vertical mappings are isomorphisms, except for the one in the middle. Now if we could prove that under these assumptions the mapping in the middle must also be an isomorphism, we will have precisely the result we want. Luckily the so-called **five lemma** tells us that this is precisely the case.

**Lemma 2.2.9.** *Suppose we have a commutative diagram*

$$\begin{tikzcd}
G_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & G_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & G_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & G_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & G_5 \arrow[d, "f_5"] \\
H_1 \arrow[r, "\beta_1"] & H_2 \arrow[r, "\beta_2"] & H_3 \arrow[r, "\beta_3"] & H_4 \arrow[r, "\beta_4"] & H_5
\end{tikzcd}$$

*of groups and group homomorphisms with exact rows. Then*
*1) If $f_1$ is surjective and $f_2, f_4$ are injective, also $f_3$ is injective.*
*2) If $f_5$ is injective and $f_2, f_4$ are surjective, also $f_3$ is surjective.*
*In particular if $f_1, f_2, f_4, f_5$ are isomorphisms, also $f_3$ is an isomorphism.*

*Proof.* We will prove 1) and leave 2) as an exercise.
The proof is an example of so-called **diagram chasing** method. Suppose $f_3(x) = 0$ for some $x \in G_3$. We must show that $x = 0$. Now

$$0 = \beta_3(f_3(x)) = f_4(\alpha_3(x)).$$

Since $f_4$ is injective, $\alpha_3(x) = 0$. Since the upper row is exact, $x = \alpha_2(y)$ for some $y \in G_2$. We have

$$\beta_2(f_2(y)) = f_3(\alpha_2(y)) = f_3(x) = 0,$$

hence, since the lower row is exact, there is $z \in H_1$ such that $\beta_1(z) = f_2(y)$. Now $f_1$ is surjective, so there is $u \in G_1$ such that $f_1(u) = z$. Consequently

$$f_2(\alpha_1(u)) = \beta_1(f_1(u)) = \beta_1(z) = f_2(y).$$

Since $f_2$ is injective this implies that $y = \alpha_1(u)$. Hence

$$x = \alpha_2(y) = \alpha_2(\alpha_1(u)) = 0$$

by exactness. ☐

We will see many examples of the applications of the five lemma.

Finally we discuss the splitting of the exact sequence.
Suppose $0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$ and $0 \longrightarrow A \xrightarrow{f'} C' \xrightarrow{g'} B \longrightarrow 0$ are short exact sequences with the same first and third group. We say that these sequences are **isomorphic (in the strong sense)** if there exists a homomorphism $\alpha\colon C \to C'$ such that the diagram

$$\begin{tikzcd}
 & & C \arrow[dd, "\alpha"] \arrow[dr, "g"] & & \\
0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "f'"'] & & B \arrow[r] & 0 \\
 & & C' \arrow[ur, "g'"'] & &
\end{tikzcd}$$

commutes. Observe that we can also write this diagram in the form

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\mathrm{id}"] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & C \arrow[r, "g"] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[d, "\mathrm{id}"] \\
0 \arrow[r] & A \arrow[r, "f"] & C' \arrow[r, "g'"] & B \arrow[r] & 0.
\end{tikzcd}$$

Since identity mappings are obviously isomorphisms, the application of Lemma 2.2.9 implies that in this case $\alpha$ must be an isomorphism. This explains the choice of the terminology.

Suppose $A$ and $B$ are abelian groups. We can always form the direct sum $A \oplus B$. There are canonical inclusions $i\colon A \to A \oplus B, j\colon A \oplus B$ defined by

$$i(a) = (a, 0),$$

$$j(b) = (0, b)$$

and canonical projections $p\colon A \oplus B \to A$, $q\colon A \oplus B \to B$ defined by

$$p(a, b) = a,$$

$$q(a, b) = b.$$

It is easy to see that the sequence

$$0 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{q} B \longrightarrow 0,$$

is short exact. We shall call it **a trivial short exact sequence** .

**Definition 2.2.10.** *Suppose*

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

*is a short exact sequence. We say that this sequence* **splits** *if it is isomorphic (in the strong sense) to the trivial sequence*

$$0 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{q} B \longrightarrow 0.$$

In practice one usually uses other, alternative definitions, presented in the next lemma.

**Lemma 2.2.11.** *Suppose*

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

*is a short exact sequence of abelian groups. Then the following conditions are equivalent.*
*1) The sequence splits.*
*2) There is a homomorphism* $f' \colon C \to A$ *such that* $f' \circ f = \mathrm{id}$.
*3) There is a homomorphism* $g' \colon B \to C$ *such that* $g \circ g' = \mathrm{id}$.

*Proof.* Suppose sequence splits, then there is an isomorphism $\alpha \colon C \to A \oplus B$ such that

$$\begin{tikzcd}
 & & C \arrow[dd, "\alpha"] \arrow[dr, "g"] & & \\
0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "i"'] & & B \arrow[r] & 0 \\
 & & A \oplus B \arrow[ur, "q"'] & &
\end{tikzcd}$$

Define $f' = p \circ \alpha \colon C \to A, g' = \alpha^{-1} \circ j$. Then

$$f'(f(a)) = p\alpha f(a) = pi(a) = p(a,0) = a,$$

$$g(g'(b)) = g\alpha^{-1}j = qj(b) = q(0,b) = b,$$

hence $f' \circ f = \mathrm{id}$ and $g \circ g' = \mathrm{id}$.
Hence 1) implies 2) and 3).

Suppose $f' \colon C \to A$ is such that $f' \circ f = \mathrm{id}$. Define $\alpha \colon C \to A \oplus B$ by

$$\alpha(c) = (f'(c), g(c)).$$

Then $q\alpha(c) = g(c)$, i.e. $q \circ \alpha = g$. Also $\alpha f(a) = (f'(f(a), g(f(a)) = (a, 0)$, hence $\alpha \circ f = i$. In other words the diagram

$$\begin{tikzcd} & & C \arrow[dd, "\alpha"] \arrow[dr, "g"] & & \\ 0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "i"'] & & B \arrow[r] & 0 \\ & & A \oplus B \arrow[ur, "q"'] & & \end{tikzcd}$$

commutes.
The proof that 3) implies 1) is similar and is left to the reader. ☐

**Lemma 2.2.12.** *Suppose*

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

*is a short exact sequence. If $B$ is a free abelian group, then this sequence splits.*

*Proof.* Exercise 2.27. ☐

As an example suppose $(X, A)$ is a topological pair and $(K, L)$ is a pair of $\Delta$-complexes. Since $C_n(X, A)$ ($C_n(K, L)$) is a free abelian group for every $n \in \mathbb{N}$, it follows that the sequences

$$0 \longrightarrow C_n(A) \xrightarrow{i_\sharp} C_n(X) \xrightarrow{j_\sharp} C_n(X, A) \longrightarrow 0$$

$$0 \longrightarrow C_n(L) \xrightarrow{i_\sharp} C_n(K) \xrightarrow{j_\sharp} C_n(K, L) \longrightarrow 0$$

split for every $n \in \mathbb{N}$.

## 2.3 Exercises

### 2.3.1 Boundary operator

1. Go through all the permutations of the set $\{0, 1, 2\}$ and make sure that the orientation of the triangle $[v_0, v_1, v_2]$ does change or does not change according to the oddity of the permutation.

2. Go through all the pairs of the 2-faces of a tetrahedron $[v_0, v_1, v_2, v_3]$ and check that their orientations are the same or not the same according to the oddity of their indices. Draw pictures!

3. Show that $\mathbb{Z}^{(A)}$ is a subgroup of $\mathbb{Z}^A$ for any set $A$.

4. Suppose $G$ is an abelian group that has a basis $A \subset G$. Show that for every mapping (of sets) $f \colon A \to H$, where $H$ is an abelian group, there exists a unique group homomorphism $g \colon G \to H$ which is an extension of $f$ i.e. $g(a) = f(a)$ for all $a \in A$.

5. Let $A$ be a set. For every $a \in A$ define $f_a \colon A \to \mathbb{Z}$ by

$$f_a(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise .} \end{cases}$$

Prove that the set $\{f_a \mid a \in A\}$ is a basis of the abelian group $\mathbb{Z}^{(A)}$.

6. Suppose $A$ is a set and $(F_A, i)$ as well as $(F'_A, i')$ are both free abelian groups on the set $A$. Show that then there exists (unique) group isomorphism $g \colon F_A \to F'_A$ such that $g \circ i = i'$.

7. Prove that a free abelian group $G$ is torsion-free i.e. for every $g \in G$ and $n \in \mathbb{N}$ the equation

$$ng = 0$$

is true if and only if $n = 0$ or $g = 0$.
Conclude that $\mathbb{Z}_n$, $n \in \mathbb{N}$ or $\mathbb{Q}/\mathbb{Z}$ are not free abelian.

8. Suppose $A \subset \mathbb{Q}$ contains at least 2 points. Show that $A$ is not independent.
Conclude that $\mathbb{Q}$ is not a free abelian group, although it is torsion-free.

9. a) Let $G$ be an abelian group and denote

$$2G = \{2g \mid g \in G\}.$$

Prove that $2G$ is a subgroup of $G$ and show that if $G \cong H$, then $G/2G \cong H/2H$.
b) Suppose $A$ and $B$ are sets, $A$ is finite. Prove that $\mathbb{Z}^{(A)} \cong \mathbb{Z}^{(B)}$ if and only if $B$ is finite and has the same amount of elements as $A$ (Hint: use a).

10. Suppose $X$ is a topological space, $n > 1$ and $0 \le j < i \le n$. Prove that

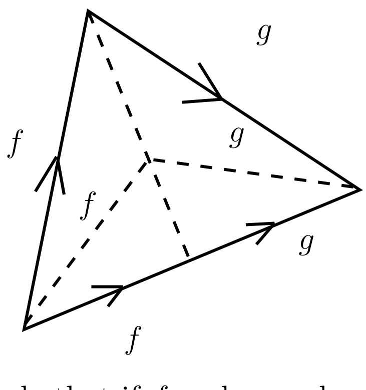
$$\partial^j_{n-1}(\partial^i_n f) = \partial^{i-1}_{n-1}(\partial^j_n f)$$

for all $f \in \mathrm{Sing}_n(X)$.

11. Suppose $X$ is a topological space. Singular 1-simplices in $X$ are mappings $f\colon I = [0,1] \to X$ and are also called **pathes** in $X$. If $f(0) = f(1)$ the path $f$ is called **the loop**. Show that as an element of $C_1(X)$ the path $f$ is a cycle if and only if it a loop.
Suppose $f, g\colon I \to X$ are pathes and $g(0) = f(1)$. Then we can defined their **product** $fcdotg\colon I \to X$ by

$$(f \cdot g)(t) = \begin{cases} f(2t), & \text{if } 0 \le t \le 1/2, \\ g(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

Prove that in this case $f + g - f \cdot g$ is a boundary element in $C_1(X)$ by constucting the explicit 2-simplex in $X$, whose boundary is $f + g - f \cdot g$. (Hint: see the picture below.)




Conclude that if $f$ and $g$ are loops, then

$$[f] + [g] = [f \cdot g] \in H_1(X).$$

12. a) Suppose $f\colon X \to Y$ is a continuous mapping. Show that $f_\sharp\colon C(X) \to C(Y)$ defined on generators by $f_\sharp(\sigma) = f \circ \sigma$ is a chain mapping.
b) Let $g\colon Y \to Z$ be another continous mapping. Prove that

$$g_\sharp \circ f_\sharp = (g \circ f)_\sharp,$$

$$\mathrm{id}_\sharp = \mathrm{id}\,.$$

Deduce the same for mappings between singular homology groups.

13. Suppose $f\colon C \to D$ is a chain mapping between chain complexes. Prove that
1) the collection of groups $\operatorname{Ker} f_n \subset C_n$ form a subcomplex of $C$, denoted by $\operatorname{Ker} f$,
2) the collection of groups $\operatorname{Im} f_n \subset D_n$ form a subcomplex of $D$, denoted by $\operatorname{Im} f$.
Deduce that $f$ induces a chain isomorphism $C/\operatorname{Ker} f \cong \operatorname{Im} f$ of chain complexes.

### 2.3.2 Simplicial homology calculations

14. a)Let $G$ be a free group on 3 free generators $a, b, c$. Show that $\{c - a, b - a, a\}$ is also a basis of $G$.
b) Let $G$ be a free group on 4 free generators $a, b, c, d$. Prove that the set $\{a + c + d, b - a + d, d\}$ is independent.

15. Consider $S^1$ as a polyhedron of a $\Delta$-complex, generated by a single 1-simplex, whose vertices are identified. Calculate the simplicial homology of this complex.
Verify that the result is the same as calculated in the lectures using the triangulation of $S^1$ as a boundary of a 2-simplex.

16. Suppose $K$ is a $\Delta$-complex. Prove that

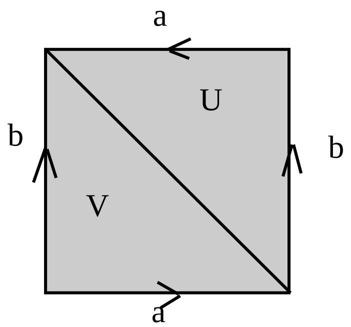
$$H_m(K^n, K^{n-1}) \cong \begin{cases} \mathbb{Z}^{(A)}, & \text{if } m = n, \\ 0, & \text{otherwise,} \end{cases}$$

where $A$ is the set of (geometrical) $n$-simplices of $K$.
Also show that the classes $[\sigma] \in H_n(K^n, K^{n-1})$, where $\sigma \in A$ form a basis of the group $H_n(K^n, K^{n-1})$.

17. Suppose $\{\alpha, \beta\}$ is a basis of a group $G$. Prove that $\{\alpha \pm \beta, \beta\}$ is also a basis of $G$.

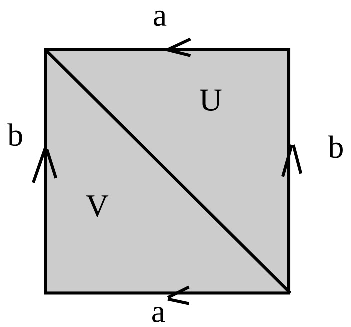
18. Suppose $G$ is a free group with basis $\{a, b, c\}$. Prove that the sets $\{c - b + a, c - a + b\}$ and $\{a - b, c\}$ are independent.

19 Calculate the homology groups of the Klein bottle, using the familiar $\Delta$-complex structure given on the picture above.

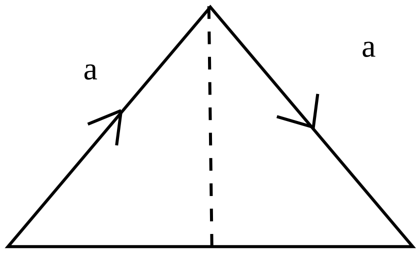



Remember to order simplices!

20. Calculate the homology groups of the torus, using the $\Delta$-complex structure




21. Calculate the homology groups of the Mobius band using the $\Delta$-complex structure as in exercise 1.25, having one triangle with two sides indentified.



Check that you end up with the same result as in the lecture notes (where we used completely different triangulation of the Mobius band).

### 2.3.3 Homological algebra

22. Find two short exact sequences of the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} C \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} C' \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0,$$

where $C$ and $C'$ are not isomorphic as groups.

23. a) Suppose

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, "f"] \arrow[d, "\alpha"] & C \arrow[r, "g"] \arrow[d, "\beta"] & \overline{C} \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & D' \arrow[r, "f'"] & D \arrow[r, "g'"] & \overline{D} \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of chain complexes and chain mappings with exact rows.
Prove that the diagram

$$\begin{tikzcd} H_n(\overline{C}) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_{n-1}(C') \arrow[d, "\alpha_*"] \\ H_n(\overline{D}) \arrow[r, "\partial"] & H_{n-1}(D') \end{tikzcd}$$

is commutative. Here $\partial$ are boundary operators induced by the horizontal short exact sequences.
b) Prove that the long exact sequence in homology is natural i.e. the diagram

$$\begin{tikzcd} \ldots \arrow[r] & H_{n+1}(\overline{C}) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_n(C') \arrow[r, "f_*"] \arrow[d, "\alpha_*"] & H_n(C) \arrow[r, "g_*"] \arrow[d, "\beta_*"] & H_n(\overline{C}) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_{n-1}(C') \arrow[r] \arrow[d, "\alpha_*"] & \ldots \\ \ldots \arrow[r] & H_{n+1}(\overline{D}) \arrow[r, "\partial"] & H_n(D') \arrow[r, "f_*"] & H_n(D) \arrow[r, "g_*"] & H_n(\overline{D}) \arrow[r, "\partial"] & H_{n-1}(D') \arrow[r] & \ldots \end{tikzcd}$$

is commutative.

24. Consider the sequence

$$\ldots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\partial} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\partial} H_{n-1}(C') \longrightarrow \ldots$$

a) Prove that

$$\operatorname{Ker}\Delta \subset \operatorname{Im} g_*.$$

b) Prove the exactness at $H_n(C')$.

25. Suppose $(X, A, B)$ is a topological triple.
a)Prove that

$$0 \longrightarrow C(A,B) \xrightarrow{i_\sharp} C(X,B) \xrightarrow{j_\sharp} C(X,A) \longrightarrow 0,$$

where $i\colon (A,B) \to (X,B)$ and $j\colon (X,B) \to (X,A)$ are obvious inclusions, is a short exact sequence. Deduce the existence of the long exact sequence

$$\ldots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial'} H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \longrightarrow \ldots.$$

b) Show that for the boundary operators of the long exact homology sequences of the pair $(X, A)$ and of the triple $(X, A, B)$ there is a commutative diagram

$$\begin{tikzcd} & H_n(A) \arrow[dd, "i_*"] \\ H_{n+1}(X,A) \arrow[ur, "\partial"] \arrow[dr, "\partial'"'] & \\ & H_n(A,B), \end{tikzcd}$$

where $i\colon A \to (A, B)$ is an inclusion. (Hint: naturality of the long exact homology sequence.)

26. Prove the second part of the Five-Lemma: Suppose the diagram of groups and homomorphisms

$$\begin{tikzcd} G_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & G_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & G_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & G_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & G_5 \arrow[d, "f_5"] \\ H_1 \arrow[r, "\beta_1"] & H_2 \arrow[r, "\beta_2"] & H_3 \arrow[r, "\beta_3"] & H_4 \arrow[r, "\beta_4"] & H_5 \end{tikzcd}$$

is commutative, rows are exact, $f_5$ is injective and $f_2, f_4$ are surjective. Then $f_3$ is surjective.

27. Suppose $H$ is a free abelian group, $G$ is an abelian group and $f\colon G \to H$ is a surjective homomorphism. Prve that there is a homomorphism $f'\colon H \to G$ such that $f \circ f' = \mathrm{id}$. Deduce that every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0,$$

where $B$ is free abelian, splits.