Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 12
12.12-16.12.2011

1. Suppose $(X, \mathcal{A})$ is a CW-complex and $(X_i, \mathcal{A}_i)$, $i \in I$ is a collection of subcomplexes of X. Prove that $(\bigcup_{i\in I} X_i, \bigcup_{i\in I} \mathcal{A}_i)$ and $(\bigcap_{i\in I} X_i, \bigcap_{i\in I} \mathcal{A}_i)$ are both subcomplexes of X.

2. a) Suppose X is a CW-complex and A is a path-component of X. Prove that A is a subcomplex of X.

   b) Suppose X is a CW-complex. Prove that the following claims are equivalent:
   1) X is connected.
   2) X is path-connected.
   3) X^1 is path-connected.
   4) Every two vertices in X^0 can be joined by a path that lies in X^1.

3. Suppose K is a simplicial complex and a, b are vertices of K. **An edge-path** from a to b is a finite sequence of vertices $a = a_0, \ldots, a_n = b$ of K such that for all $i = 0, \ldots, n$ a_i and a_{i+1} belong to the same 1-simplex τ_i. In this case also the sequence $\tau_0, \ldots, \tau_{n-1}$ is also called an edge-path from a to b.
   Prove that $|K|$ is connected if and only if for every pair of vertices $a, b \in K$ there is an edge-path from a to b.

4. Suppose $g \in \mathbb{N}$ ($g \geq 1$). Show that M_g (N_g) is a connected compact 2-manifold without boundary, which can be triangulated.

5. Suppose K is a 2-dimensional simplicial complex and $\tau \in K$ is a 1-simplex which is a face of exactly n 2-simplices. Suppose x is an interior point of τ. Prove that
$$H_1(|Lk(x)|) \cong \mathbb{Z}^{n-1}.$$

6. Suppose K is a finite simplicial complex such that $|K|$ is an n-dimensional manifold, possibly with boundary. Prove that $|Lk(x)|$ has the homotopy type of S^{n-1}, if $x \in |K|$ is an interior point and contractible if x is the boundary point.
   Assuming $n = 2$ prove that $|K|$ is 2-dimensional as simplicial complex and every 1-simplex of K is a face of two or one 2-simplex. Moreover if L is a subcomplex of K generated by 1-simplices that are faces of exactly one 2-simplex, then $|L| = \partial|K|$.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.