Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 10
21.11-25.11.2011

1. Suppose $(X; U, V)$ is a proper triad such that $U \cap V \neq \emptyset$. Prove the existence of the reduced exact Mayer-Vietoris sequence

$$\dots \longrightarrow \widetilde{H}_{n+1}(X) \xrightarrow{\partial} \widetilde{H}_n(U \cap V) \xrightarrow{i_*} \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \xrightarrow{j_*} \widetilde{H}_n(X) \xrightarrow{\partial} \widetilde{H}_{n-1}(U \cap V) \longrightarrow \dots$$

(Hint: ordinary Mayer-Viatoris+Lemma 3.1.7).

2. Construct the explicit formula for the mapping g defined in the proof of the Brouwer's fixed point Theorem (theorem 3.4.6) and show that g is continuous retract $\overline{B}^n \to S^{n-1}$.

3. a) Suppose V is an open subset of $\mathbb{R}^n, n \geq 2$ and $x \in V$. Using excision property show that $H_1(V, V \setminus \{x\}) \cong H_1(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ and deduce that $H_1(V, V \setminus \{x\}) = 0$.
Using this, prove that $V \setminus \{x\}$ is path-connected, if V is path-connected.
b) Suppose $n \geq 2$ and $S \subset \mathbb{R}^n$ is homeomorphic to S^{n-1}.
Prove that $\mathbb{R}^n \setminus S$ has exactly two path components U and V, where U is bounded, V is not and $S = \partial U = \partial V$.
What happens if $n = 1$?

4. Suppose U is an open subset of $\mathbb{R}^n$ and $f \colon U \to \mathbb{R}^n$ is a continuous injection. Prove that f is open, in particular $V = f(U)$ is open and $f \colon U \to V$ is a homeomorphism.
(Hint: Invariance of Domain and local compactness of $\mathbb{R}^n$.)

5. Suppose M is an m-manifold, N is an n-manifold. Prove that
1) If $m > n$ there are no continuous injections $M \to N$.
2) If $m = n$ and M has no boundary, then any continuous injection $f \colon M \to N$ is an open embedding, i.e. a homeomorphism to the image $f(M)$, which is open in N (and is a subset of $\operatorname{int} M$).
3) If $M \cong N$, then $m = n$.

6. Suppose M is an n-manifold. Prove that
1) The sets ∂M and $\operatorname{int} M$ are disjoint.
2)int M is open in M and itself is an n-manifold without boundary.
3)∂M is closed in M and is an $(n-1)$-manifold without boundary (if non-empty).

7. Let M be a Mobius band. Prove that M is a manifold with boundary and $\partial M \cong S^1$. What is the dimension of M as a manifold?
Let $i \colon \partial M \hookrightarrow M$ be inclusion. Prove that $i_* \colon H_1(\partial M) \to H_1(M)$ is essentially a homomorphism $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto 2n$. Conclude that ∂M is not a retract of M.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.