Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 10
Solutions

1. Suppose $(X; U, V)$ is a proper triad such that $U \cap V \neq \emptyset$. Prove the existence of the reduced exact Mayer-Vietoris sequence

$$\ldots \longrightarrow \widetilde{H}_{n+1}(X) \xrightarrow{\partial} \widetilde{H}_n(U \cap V) \xrightarrow{i_*} \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \xrightarrow{j_*} \widetilde{H}_n(X) \xrightarrow{\partial} \widetilde{H}_{n-1}(U \cap V) \longrightarrow \ldots$$

Solution: The following diagram is commutative.

$$\begin{array}{ccccccccccc}
\ldots & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & \widetilde{H}_1(X) & \xrightarrow{\partial} & \widetilde{H}_0(U \cap V) & \xrightarrow{i_*} & \widetilde{H}_0(U) \oplus \widetilde{H}_0(V) & \xrightarrow{j_*} & \widetilde{H}_0(X) & \xrightarrow{\partial} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & H_1(X) & \xrightarrow{\partial} & H_0(U \cap V) & \xrightarrow{i_*} & H_0(U) \oplus H_0(V) & \xrightarrow{j_*} & H_0(X) & \xrightarrow{\partial} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i,-i} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\mathrm{id}+\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & & 0 & & 0 & & 0 & & 0 & & 0.
\end{array}$$

Moreover all columns are short exact sequences, middle and lower rows are also exact. It follows (Lemma 3.1.7) that also the upper row is exact.

2. Construct the explicit formula for the mapping g defined in the proof of the Brouwer's fixed point Theorem (theorem 3.4.6) and show that g is continuous retract $\overline{B}^n \to S^{n-1}$.

Solution: Let $f\colon \overline{B}^n \to \overline{B}^n$ be a continuous mapping without fixed points. For every $x \in \overline{B}^n$ consider a half-line

$$L_x = \{f(x) + t(x - f(x)) \mid t \geq 0\}$$

and let $g(x)$ be the unique point $L_x \cap S^{n-1}$. Let us show that g is well-defined and continuous. Now $y = f(x) + t(x - f(x)) \in S^{n-1}$ if and only if

$$|y|^2 = |f(x)|^2 + 2t < x - f(x), f(x) > +t^2|x - f(x)|^2 = 1.$$

This equation has solutions

$$t(x) = (-2 < x - f(x), f(x) > \pm\sqrt{4 < x - f(x), f(x) >^2 +4|x - f(x)|^2(1 - |f(x)|^2)})/(2|x - f(x)|^2)$$

The expression under square root is always even strictly positive, since $|f(x)| \leq 1$ for all $x \in \overline{B}^n$ and $|x - f(x)| > 0$ by assumption. Moreover it follows that

only solution with plus sign satisfies condition $t(x) > 0$.
Hence we obtain formula

$$g(x) = f(x) + t(x)(x - f(x)), \text{where}$$

$$t(x) = \sqrt{< x - f(x), f(x) >^2 + |x - f(x)|^2(1 - |f(x)|^2)} - < x - f(x), f(x) >)/|x - f(x)|^2.$$

Clearly $g\colon \overline{B}^n \to S^{n-1}$ is then continuous well-defined mapping. It remains to check that $g(x) = x$ for all $x \in S^{n-1}$. It is enough to show that $t(x) = 1$ in this case. If $|x| = 1$, then

$$|x - f(x)|^2 = |x|^2 - 2 < x, f(x) > + |f(x)|^2 = 1 - 2 < x, f(x) > + |f(x)|^2,$$

hence

$$< x - f(x), f(x) >^2 + |x - f(x)|^2(1 - |f(x)|^2) =$$

$$= < x, f(x) >^2 + |f(x)|^4 - 2 < x, f(x) > |f(x)|^2 + 1 - 2 < f(x), x > + |f(x)|^2 - |f(x)|^2 + 2 < f(x), x >$$

$$= < x, f(x) >^2 - 2 < f(x), x > + 1 = (1 - < x, f(x) >)^2.$$

By Cauchy's inequality

$$< x, f(x) > \leq |x| \cdot |f(x)| \leq 1,$$

hence

$$t(x) = (1 - < x, f(x) >) - < x, f(x) > + |f(x)|^2)/(1 - 2 < x, f(x) > + |f(x)|^2) = 1.$$

3. a) Suppose V is an open subset of $\mathbb{R}^n, n \geq 2$ and $x \in V$. Using excision property show that $H_1(V, V \setminus \{x\}) \cong H_1(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ and deduce that $H_1(V, V \setminus \{x\}) = 0$.
Using this, prove that $V \setminus \{x\}$ is path-connected, if V is path-connected.
b) Suppose $n \geq 2$ and $S \subset \mathbb{R}^n$ is homeomorphic to S^{n-1}.
Prove that $\mathbb{R}^n \setminus S$ has exactly two path components U and V, where U is bounded, V is not and $S = \partial U = \partial V$.
What happens if $n = 1$?

Solution: a) Let $A = \mathbb{R}^n \setminus V$ and $W = \mathbb{R}^n \setminus \{x\}$. Then the closure of A is contained in the interior of W, hence by excision property $H_1(\mathbb{R}^n \setminus A, W \setminus A) \cong H_1(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$. But

$$(\mathbb{R}^n \setminus A, W \setminus A) = (V, V \setminus \{x\}),$$

so the first claim follows.

To show that $H_1(V, V \setminus \{x\}) = 0$ it is sufficient to show that $H_1(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) = 0$. But

$$H_1(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \widetilde{H}_0(\mathbb{R}^n \setminus \{x\}) = 0,$$

by the reduced long exact homology sequence of the pair $\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}$ and the fact that $\mathbb{R}^n$ has trivial reduced groups. $\mathbb{R}^n \setminus \{x\}$ is path-connected since $n > 1$.

Next suppose V is path-connected. Then $\widetilde{H}_0(V) = 0$. Since $H_1(V, V \setminus \{x\}) = 0$, reduced long exact homology sequence of the pair $(V, V \setminus \{x\})$ implies that $\widetilde{H}_0(V \setminus \{x\}) = 0$, hence $V \setminus \{x\}$ is path-connected.

b) Since $\mathbb{R}^n$ is homeomorphic to the subspace $S^n \setminus \{e_{n+1}\}$ of S^n, we can consider S a subspace of S^n. Jordan-Brouwer separation theorem 3.6.4 $S^n \setminus S$ has exactly two path-components U and W, where we may assume that $e_{n+1} \in W$.

Also $\partial U = \partial V = S$, where boundary is taken in S^n.

By stereographic projection W is homeomorphic to a subset of $\mathbb{R}^n$, hence by a) $V = W \setminus \{e_{n+1}\}$ is path-connected. Also both sets U and V are open in S^n, hence also in $\mathbb{R}^n$, so it follows that U and V are path-components of $\mathbb{R}^n \setminus S$.

Since both U and V are open we have

$$\partial_{\mathbb{R}^n}(U) = \mathrm{cl}_{\mathbb{R}^n}(U) \setminus U,$$
$$\partial_{\mathbb{R}^n}(V) = \mathrm{cl}_{\mathbb{R}^n}(V) \setminus V.$$

Also

$$\mathrm{cl}_{\mathbb{R}^n}(U) = \mathrm{cl}_{S^n}(U) \cap \mathbb{R}^n = (U \cup S) \cap \mathbb{R}^n = U \cup S,$$
$$\mathrm{cl}_{\mathbb{R}^n}(V) = \mathrm{cl}_{S^n}(V) \cap \mathbb{R}^n.$$

It follows immediately that

$$\partial_{\mathbb{R}^n}(U) = (U \cup S) \setminus U = S.$$

Also

$$\mathrm{cl}_{S^n}(V) = \mathrm{cl}_{S^n}(W).$$

To see this it enough to prove that $e_{n+1} \in \mathrm{cl}_{S^n}(V)$. Let A be a neighbourhood of e_{n+1}. Then $A \cap W$ is a neighbourhood of e_{n+1} and since S^n is not discrete, there is a point $x \in A \cap W, x \neq e_{n+1}$, hence A contains a point from V. Hence we obtain

$$\mathrm{cl}_{\mathbb{R}^n}(V) = \mathrm{cl}_{S^n}(W) \cap \mathbb{R}^n = (W \cup S) \cap \mathbb{R}^n = V \cup S, \text{ so}$$
$$\partial_{\mathbb{R}^n}(V) = (V \cup S) \setminus V = S.$$

It remains to show that one of the components of $\mathbb{R}^n \setminus S$ is bounded and the other one is not. Let $R > 0$ be big enough so that $S \subset B(0, R)$. Since $n > 1$, the set $\mathbb{R}^n \setminus B(0, R)$ is path-connected and does not intersect S, hence it is contained entirely in one of the components, say V, which then must be unbounded. It follows that in this case $U \subset B(0, R)$, so U must be bounded.

If $n = 1$ the Jordan-Brouwer separation theorem is not true in $\mathbb{R}^1$. Indeed a subset of $\mathbb{R}$ which is homeomorphic to S^0 is just two points, so $\mathbb{R} \setminus S$ has exactly 3 path-components. Also S is then a boundary of only one of them.

4. Suppose U is an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^n$ is a continuous injection. Prove that f is open, in particular $V = f(U)$ is open and $f\colon U \to V$ is a homeomorphism.

Solution: It is enough to show that f is locally open, i.e. every point $x \in U$ has a neighbourhood $W \subset U$ such that $f|W$ is open.

Since $\mathbb{R}^n$ is locally compact, we can choose $W \ni x$ such that $\overline{W} \subset U$ is compact. Then $f|\overline{W}$ is a continuous injection from compact space to the Hausdorff space $\mathbb{R}^n$, hence is an embedding. It follows that also $f|W$ is an embedding as well. Now Invariance of Domain (Theorem 3.6.5) implies that for every open $A \subset W$ $f(A)$ is open, hence $f|W$ is an open mapping.

5. Suppose M is an m-manifold, N is an n-manifold. Prove that
1) If $m > n$ there are no continuous injections $M \to N$.
2) If $m = n$ and M has no boundary, then any continuous injection $f\colon M \to N$ is an open embedding, i.e. a homeomorphism to the image $f(M)$, which is open in N (and is a subset of $\operatorname{int} M$).
3) If $M \cong N$, then $m = n$.

Solution: c) follows directly from a). To prove a) take $x \in \operatorname{int} M$ and let V be an open subset of N, which contains $f(x)$ and is homeomorphic to an open subset of $\mathbb{H}_n$. Since f is continuous, there exists open neighbourhood U of x in M such that $f(U) \subset V$. We may assume that U is homeomorphic to an open subset of $\mathbb{R}^m$. Hence $f|U$ defines a continuous injection $U \to \mathbb{R}^n$. Since $\mathbb{R}^n$ can be thought of as a subset of $\mathbb{R}^m$ defined by

$$\mathbb{R}^n = \{(x_1, \dots, x_n, \dots, x_m) \in \mathbb{R}^m \mid x_{n+1} = \dots = x_m\}$$

we obtain a continuous injection $g\colon U \to \mathbb{R}^m$. By the previous exercise $f(U)$ is open in $\mathbb{R}^m$. But this cannot be true, since $f(U)$ is a subset of $\mathbb{R}^n$, which has no interior points with respect to $\mathbb{R}^m$.

To prove b) it is enough to prove that f is locally open i.e. every point $x \in M$ has an open neighbourhood U such that $f|U$ is open mapping. We can choose U to be a neighbourhood of x which is homeomorphic to an open subset of $\mathbb{R}^n$ such that $f(U) \subset V$, where V is open in N and homeomorphic to a subset of $\mathbb{R}^n$. Previous exercise then implies that $f|U$ is open embedding.

6. Suppose M is an n-manifold. Prove that
1) The sets ∂M and $\operatorname{int} M$ are disjoint.
2)$\operatorname{int} M$ is open in M and itself is an n-manifold without boundary.
3)∂M is closed in M and is an $(n-1)$-manifold without boundary (if non-empty).

Solution: 1) Suppose $x \in \partial M \cap \operatorname{int} M$. Then there exists open neighbourhoods U, V of x and homeomorphisms $f\colon U \to f(U) \subset \mathbb{R}^n$ and $g\colon V \to f(V) \subset \mathbb{H}^n$, where $f(U)$ is open in $\mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n-1}$. Then $f|U \cap V \to f(U \cap V)$ and $g|U \cap V \to g(U \cap V)$ and homeomorphisms, where $f(U \cap V)$ is open in $\mathbb{R}^n$ and $g|U \cap V(x) \in \mathbb{R}^{n-1}$, so we may assume $U = V$.
Now $g \circ f^{-1}\colon f(U) \to g(U)$ is a homeomorphism between open subset of $\mathbb{R}^n$ and non-open subset $g(U)$ of $\mathbb{R}^n$. This contradicts Invariance of Domain (Theorem 3.6.5).

2) Suppose $x \in \operatorname{int} M$ and let $f\colon U \to f(U)$ be a homeomorphism, $U \ni x$ open in M and $f(U)$ open in $\mathbb{R}^n$. Then by definition $U \subset \operatorname{int} M$, so $\operatorname{int} M$ is open in M. Also f serves as a chart for x in $\operatorname{int} M$, so $\operatorname{int} M$ is n-manifold without boundary (it is always non-empty, since M has at least one chart).

3) Since by 1)

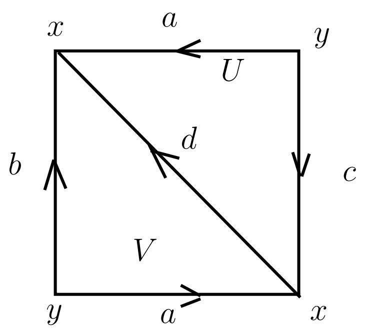
$$\partial M = M \setminus \operatorname{int} M,$$

2) implies that ∂M is closed in M. Suppose $x \in \partial M$ and let $f\colon U \to f(U)$ be a homeomorphism,where $U \ni x$ open in M and $f(U)$ open in $\mathbb{H}^n$ and $f(x) \in \mathbb{R}^{n-1}$. It follows that

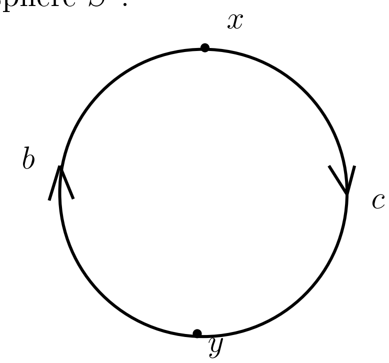
$$U \cap \partial M = f^{-1}(\mathbb{R}^{n-1}),$$

so the restriction $f|\colon U \cap \partial M \to \mathbb{R}^{n-1}$ is a continuous embedding, where $f|(U \cap \partial M) = f(U) \cap \mathbb{R}^{n-1}$ is open in $\mathbb{R}^{n-1}$. It follows that ∂M is an $n-1$-manifold without boundary.

7. Let M be a Mobius band. Prove that M is a manifold with boundary and $\partial M \cong S^1$. What is the dimension of M as a manifold?
Let $i\colon \partial M \hookrightarrow M$ be inclusion. Prove that $i_*\colon H_1(\partial M) \to H_1(M)$ is essentially a homomorphism $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto 2n$. Conclude that ∂M is not a retract of M.
Solution: Represent Mobius band triangulated as a polyhedron of a Δ-complex with two 2-simplices, as usual.



Let L be a subcomplex generated by 1-simplices b and c. $|L|$ looks like a sphere S^1.



Using simplicial homology we see that $H_1(|L|) \cong \mathbb{Z}$ with generator $[c-b]$. On the other hand in the Example 2.1.23 we calculated that $H_1(|K|) \cong \mathbb{Z}$ with generator $[d]$. On the other hand

$$\partial U = d - a + c, \partial V = d + a - b,$$

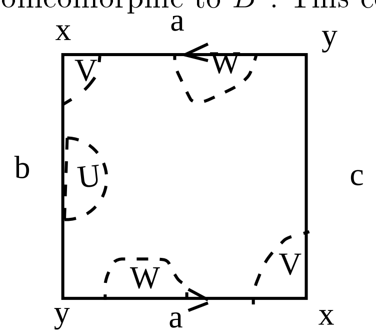
so in homology $[d] = [c-a] = [a-b]$. Hence

$$i_*([c-b]) = [c-b] = [c-a] + [a-b] = 2d,$$

so up to isomorphism $i_*\colon \mathbb{Z} \to \mathbb{Z}$ is a mapping $n \mapsto 2n$.
If there would exist retraction $r\colon |K| \to |L|$, then $r \circ i = \mathrm{id}$, so $r_* \circ i* = \mathrm{id}$, so in particular $r_*\colon \mathbb{Z} \to \mathbb{Z}$ is a surjective homomorphism. But the only surjective homomorphisms from $\mathbb{Z}$ to itself are identity mapping or mapping $n \mapsto -n$, and both are in fact bijections. Hence r_* is bijective, so it follows that $i_* = r_*^{-1}$ is bijective. But this contradicts the calculation above, since

the mapping $n \mapsto 2n$ is not surjective mapping $\mathbb{Z} \to \mathbb{Z}$.
Thus we have shown that $|L|$ is not a retract of $|K|$.

It remains to show that $|K|$ is a 2-manifold and $|L|$ is its boundary. The picture below illustrates typical neighbourhood of the points of $|K|$. U is a typical neighbourhood of the interior point of b and it is homeomorphic to the open subset of $\mathbb{H}^2$. Neighbourhoods of the points in the interior of c look the same. V is a neighbourhood of the vertex x, which in the picture consists of two parts, but after identification along side a V is a neighbourhood that is homeomorphic to the open subset of $\mathbb{H}^2$. Same works for the other vertex y. W is a neighbourhood of a point in the interior of a and it is homeomorphic to the open disk B^2. Points in the interior of a square clearly have a neighbourhood homeomorphic to B^2. This concludes the proof of the claim.



Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.