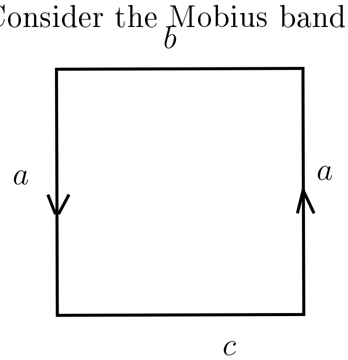
Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 9
14.11-19.11.2011

1. Prove that the singular homology **has compact carriers** in the following precise sense.
   a) Suppose $x \in H_n(X)$ (X a top. space). Prove that there exists compact $C \subset X$ such that x belongs to the image of
   $$i_*\colon H_n(C) \to H_n(X)$$
   (where $i\colon C \to X$ inclusion).
   b) Suppose $C \subset X$ is compact, $i\colon C \to X$ an inclusion and $x \in H_n(C)$ is such that $i_*(x) = 0 \in H_n(X)$. Prove that there exists a compact $D \subset X$ such that $C \subset D$ and $j_*(x) = 0 \in H_n(D)$, where $j\colon C \to D$ is inclusion.
   Also prove a) and b) for reduced homology groups $\tilde{H}_n$.

2. Suppose K is a Δ-complex.
   a) Let C be a compact subset of $|K|$. Show that there is a finite subcomplex L of K such that $C \subset L$.
   b) Assume the theorem 3.4.3 (the equivalence of simplicial and singular homologies) is true for all finite subcomplexes of K. Prove that $i_*\colon H_n(K) \to H_n(|K|)$ is an isomorphism for all $n \in \mathbb{N}$. (Hint: a) and the previous exercise).

3. Consider the Mobius band X triangulated as usual.



   a) Calculate the simplicial homology of the "boundary"i.e. a subcomplex generated by the 1-simplices a, b, c.

   b) Deduce that Mobius band and S^1 are not homeomorphic (remove a point and use b)).

4. a) Let $n > 0, i \in \{1, \dots, n+1\}$ and let $\iota_i\colon S^n \to S^n$ be defined by $\iota_i(x) = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n + 1)$. Show that that
   $$(\iota_i)_*(x) = -x$$

for all $x \in H_n(S^n)$, $i = 1, \dots, n$, assuming this is known for ι_{n+1} (proved in the lecture notes). (Hint: use the fact that $\iota_i = f \circ \iota_{n+1} \circ f$ for some homeomorphism f.)
b) Let $h\colon S^n \to S^n$, $h(x) = -x$. Prove that

$$h_*(x) = (-1)^{n+1}x.$$

for all $x \in H_n(S^n)$.

5. Suppose $D = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$ be a finite subdivision of $I = [0, 1]$. Define for every $i = 0, \dots, n-1$ a path $\alpha_i\colon I \to S^1$ by

$$\alpha_i(t) = \cos(2\pi t_i(1-t) + t2\pi t_{i+1}) + i\sin(2\pi t_i(1-t) + t2\pi t_{i+1}).$$

In other words α_i is an arc that connects $x_i = e^{2\pi t_i}$ and $x_{i+1} = e^{2\pi t_{i+1}}$. Define $\gamma_D \in C_1(S^1)$ as

$$\gamma_D = \sum_{i=0}^{n-1} \alpha_i.$$

Show that γ_D is a cycle. By induction on n prove that $[\gamma_D] = [\gamma] \in H_1(S^1)$, where $\gamma = \gamma_{D_0}$, $D = \{0, 1\}$. (Hint: exercise 2.7).

Conclude that $[\gamma_D]$ is a generator of $H_1(S^1)$ for every D.

6. a) Suppose K is a simplicial complex and L_1 and L_2 are subcomplexes of K such that $K = L_1 \cup L_2$. Show that $(|K|; |L_1|, |L_2|)$ is a proper triad. (Hint: use the equivalence of simplicial and singular homologies).

b)Show that $(S^n; B_+, B_-)$ is a proper triad using a). Write down the Mayer-Vietoris sequence of this triad and use it to prove that $H_n(S^n) \cong H_{n-1}(S^{n-1})$ for $n > 1$.

c) Can you prove that $(S^n; B_+, B_-)$ is a proper triad using the properties of the singular homology, such as homotopy axiom and Mayer-Vietoris sequence for the open covering by 2 sets?

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.