Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 9
Solutions

1. Prove that the singular homology **has compact carriers** in the following precise sense.
a) Suppose $x \in H_n(X)$ (X a top. space). Prove that there exists compact $C \subset X$ such that x belongs to the image of

$$i_*\colon H_n(C) \to H_n(X)$$

(where $i\colon C \to X$ inclusion).
b) Suppose $C \subset X$ is compact, $i\colon C \to X$ an inclusion and $x \in H_n(C)$ is such that $i_*(x) = 0 \in H_n(X)$. Prove that there exists a compact $D \subset X$ such that $C \subset D$ and $j_*(x) = 0 \in H_n(D)$, where $j\colon C \to D$ is inclusion.
Also prove a) and b) for reduced homology groups $\tilde{H}_n$.

Solution: a) Suppose $x \in H_n(X)$. Then there exist $m \in \mathbb{N}$, singular n-simplices $\sigma_i\colon \Delta_n \to X$, $i = 0, \ldots, m$ and integers $n_0, \ldots, n_m$ such that

$$x = [\sum_{i=0}^{m} n_i \sigma_i],$$

where $y = \sum_{i=0}^{m} n_i \sigma_i \in Z_n(X)$ is a cycle.
Define

$$C = \bigcup_{i=0}^{m} \sigma_i(\Delta_n).$$

Then C is a compact subset of X and by restricting image of σ_i for all $i = 0, \ldots, m$ we may regard y as an element in $C_n(C)$. Moreover y is still a cycle in C, so there is

$$x' = [y] \in H_n(C)$$

and $i_*(x') = x$.

If $x \in \tilde{H}_0(X)$, then y above has property $\varepsilon(y) = 0$, so it has the same property considered as an element of $C_0(C)$ as well. It follows that $x' = [y] \in \tilde{H}_0(C)$.

b) As above represent x as a class $[y]$, where

$$y = \sum_{i=0}^{m} n_i \sigma_i,$$

for some singular n-simplices $\sigma_i\colon \Delta_n \to C$. Since $i_*(x) = 0 \in H_n(X)$, this implies that there exists

$$z = \sum_{j=0}^{k} n_j \tau_j \in C_{n+1}(X)$$

so that $\partial(z) = y$. Let

$$D = \bigcup_{j=0}^{l} \tau_j(\Delta_{n+1}) \cup C.$$

Then D is compact and $C \subset D$. Moreover $z \in C_{n+1}(D)$ and $\partial(z) = j_\sharp(y)$ i.e. $j_\sharp(y)$ is a boundary in D. Hence $j_*(y) = 0$.

This time reduced groups don't affect conclusion in any way, so the claim is trivially true also for $\tilde{H}_0$.

2. Suppose K is a Δ-complex.
a) Let C be a compact subset of $|K|$. Show that there is a finite subcomplex L of K such that $C \subset L$.
b) Assume the theorem 3.4.3 (the equivalence of simplicial and singular homologies) is true for all finite subcomplexes of K. Prove that $i_*\colon H_n(K) \to H_n(|K|)$ is an isomorphism for all $n \in \mathbb{N}$. (Hint: a) and the previous exercise).

Solution: a) It is enough to show that C intersects $\operatorname{int}\sigma$ for finitely many $\sigma \in K$ only, since then the subcomplex L formed by these simplices and all their faces is also finite.
For every $\sigma \in K$ such that $C \cap \operatorname{int}\sigma \neq \emptyset$ choose one point $x_\sigma \in C \cap \operatorname{int}\sigma$ and consider the set

$$B = \{x_\sigma \mid \sigma \in K, C \cap \operatorname{int}\sigma \neq \emptyset\}.$$

Then $B \subset C$. It is enough to prove B is finite. Suppose $A \subset B$ and $\sigma \in K$ arbitrary. Since A intersects every interior of a simplex in at most one point and σ intersects finitely many interiors of simplices, it follows that $A \cap \sigma$ is finite, in particularly closed in σ. Since $|K|$ has weak topology, it follows that A is closed in $|K|$. In follows that every subset of B is closed, hence B has discrete topology and B itself is a closed subset of C, hence compact. Since compact and discrete space is always finite, claim follows.

b) Suppose $y \in H_n(|K|)$. By exercise 1 there exists compact $C \subset |K|$ such that $y = j_*(y')$ for some $y' \in H_n(C)$ $j\colon C \to |K|$ inclusion. By a) there exists finite subcomplex L of K such that $C \subset |L|$, so $y = k_*(y'')$, where $y'' = j'_*(y') \in H_n(L)$, $j'\colon C \to |L|$, $k\colon |L| \to |K|$ inclusions. Consider commutative diagram

$$\begin{array}{ccc}
H_n(L) & \xrightarrow{i_* \cong} & H_n(|L|) \\
\downarrow l_* & & \downarrow k_* \\
H_n(K) & \xrightarrow{i_*} & H_n(|K|),
\end{array}$$

where $l\colon L \to K$ is inclusion of Δ-complexes. Since $i_*\colon H_n(L) \to H_n(|L|)$ is surjection by assumption, there exists $x'' \in H_n(L)$ such that $i_*(x'') = y''$. Let $x = l_(x'')$. Then

$$i_*(x) = i_*(l_*(x'')) = k_*(i_*(x'')) = k_*(y'') = y.$$

Hence i_* is surjection.

Suppose $x \in H_n(K)$ such that $i_*(x) = 0 \in H_n(|K|)$. Now

$$x = [\sum_{i=1}^{m} n_i \sigma_i],$$

where $m \in \mathbb{N}$, $\sigma_i \in K$, $i = 1, \dots, m$. The subcomplex L generated by simplices $\sigma_i, i = 1, \dots, m$ and all their faces is finite and $x' = [\sum_{i=1}^m n_i\sigma_i] \in H_n(L)$ is such that $l_*(x') = x$ for inclusion $l\colon L \to K$. Let $k\colon |L| \to |K|$ be an inclusion. From the commutativity of the diagram

$$\begin{array}{ccc} H_n(L) & \xrightarrow{i_*} & H_n(|L|) \\ \downarrow{\scriptstyle l_*} & & \downarrow{\scriptstyle k_*} \\ H_n(K) & \xrightarrow{i_*} & H_n(|K|), \end{array}$$

we see that

$$k_*(i_*(x')) = i_*(l(x')) = i_*(x) = 0.$$

Since L is finite, $|L|$ is compact, so by the exercise 1b) there exists compact $D \supset |L|$ such that $k'_*(i_*(x)) = 0$ for $k\colon |L| \to D$ is an inclusion. By enlarging D to a finite subcomplex (which exists according to a)) we may assume $D = |L'|$, where L' is a finite subcomplex. We have a commutative diagram

$$\begin{array}{ccc} H_n(L) & \xrightarrow{i_* \cong} & H_n(|L|) \\ \downarrow & & \downarrow{\scriptstyle k'_*} \\ H_n(L') & \xrightarrow{i_* \cong} & H_n(|L'|) \\ \downarrow{\scriptstyle l''_*} & & \downarrow{\scriptstyle k_*} \\ H_n(K) & \xrightarrow{i_*} & H_n(|K|). \end{array}$$

Now if $l'\colon L \to L'$ denote a simplicial inclusion we have

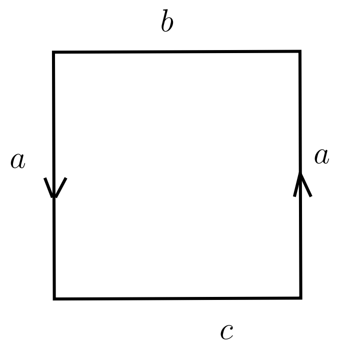
$$i_*(l'_*(x')) = k'_*(i_*(x')) = 0,$$

so by assumption $l'_*(x') = 0 \in H_n(L')$. It follows that

$$x = l_*(x') = l''_*(l'_*(x')) = 0,$$

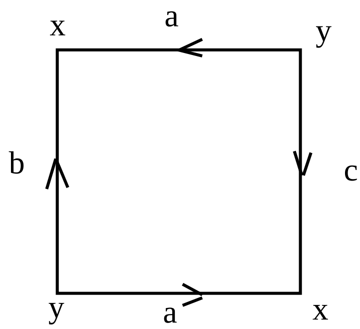
where $l''\colon L' \to K$ is a simplicial inclusion.
Hence i_* is an injection.

3. Consider the Mobius band X triangulated as usual.



a) Calculate the simplicial homology of the "boundary"i.e. a subcomplex generated by the 1-simplices a, b, c.

b) Deduce that Mobius band and S^1 are not homeomorphic (remove a point and use a)).



Solution: a)

Let L be a subcomplex generated by 1-simplices a, b, c. Since L is 1-dimensional it is enough to calculate only groups $H_1(L) = \operatorname{Ker}\partial_1$ and $H_0(L)$. There are two vertices, x and y (see the picture) and

$$\partial_1(a) = \partial_1(b) = \partial_1(c) = x - y,$$

hence

$$\partial_1(ma + nb + lc) = (m + n + l)(y - x) = 0$$

if and only if $l = -(m + n)$. It follows that $H_1(L) = \operatorname{Ker}\partial_1$ is a free abelian group on two generators $a - c$ and $b - c$, i.e. isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.
It also follows that the image $B_0(L) \subset C_0(L)$ is a subgroup generated by $x - y$. Since $\{x, x - y\}$ is a basis of $C_0(L)$, it follows that

$$H_0(L) = C_0(L)/B_0(L) \cong \mathbb{Z}[x] \cong \mathbb{Z}.$$

b)If Mobius band and S^1 are homeomorphic, then also $M \setminus \{x\}$ and $S^1 \setminus \{y\}$ are also homeomorphic, where we choose x to be an "interior point "of the square and y is the image of x under isomorphism. But $M \setminus \{x\}$ has the same homotopy type as the "boundary "$|L|$, while $S^1 \setminus \{y\}$ is contractible. We obtain contradiction, since a) implies that $H_1(M \setminus \{x\}) \cong \mathbb{Z} \oplus \mathbb{Z}$, in particular not trivial.

4. a) Let $n > 0, i \in \{1, \ldots, n+1\}$ and let $\iota_i \colon S^n \to S^n$ be defined by $\iota_i(x) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n + 1)$. Show that that

$$(\iota_i)_*(x) = -x$$

for all $x \in H_n(S^n)$, $i = 1, \ldots, n$, assuming this is known for ι_{n+1} (proved in the lecture notes). (Hint: use the fact that $\iota_i = f \circ \iota_{n+1} \circ f$ for some

homeomorphism f.)
b) Let $h\colon S^n \to S^n$, $h(x) = -x$. Prove that

$$h_*(x) = (-1)^{n+1}x.$$

for all $x \in H_n(S^n)$.

Solution: a) Let $f\colon S^n \to S^n$ be a mapping that interchages ith and $n+1$th coordinates, i.e.

$$f(x_1, \ldots, , x_{i-1}, x_i, x_{i+1}, \ldots, x_{n+1}) = (x_1, \ldots, , x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_i).$$

Mapping f is clearly a homeomorphism and

$$\iota_i = f \circ \iota_{n+1} \circ f.$$

Now $f \circ f = \mathrm{id}$, so for every $x \in H_n(S^n)$ we have

$$(\iota_i)_*(x) = f_*((\iota_{n+1})_*(f_*(x)) = f_*(-f_*(x)) = -(f_* \circ f_*)(x) = -x.$$

b)Since

$$h = \iota_1 \circ \iota_2 \circ \ldots \circ \iota_n \circ \iota_{n+1}$$

claim follows easily from a).

5. Suppose $D = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$ be a finite subdivision of $I = [0, 1]$. Define for every $i = 0, \ldots, n-1$ a path $\alpha_i\colon I \to S^1$ by

$$\alpha_i(t) = \cos(2\pi t_i(1-t) + t2\pi t_{i+1}) + i\sin(2\pi t_i(1-t) + t2\pi t_{i+1}).$$

In other words α_i is an arc that connects $x_i = e^{2\pi t_i}$ and $x_{i+1} = e^{2\pi t_{i+1}}$. Define $\gamma_D \in C_1(S^1)$ as

$$\gamma_D = \sum_{i=0}^{n-1} \alpha_i.$$

Show that γ_D is a cycle. By induction on n prove that $[\gamma_D] = [\gamma] \in H_1(S^1)$, where $\gamma = \gamma_{D_0}$, $D = \{0, 1\}$. (Hint: exercise 4.7).

Conclude that $[\gamma_D]$ is a generator of $H_1(S^1)$ for every D.

Solution: Suppose $n > 1$ and $D = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$ be a finite subdivision of $I = [0, 1]$. Then $0 < t_{n-1} < 1$. Let $D' = \{0 = t_0 < t_1 < \ldots < t_{n-2} < t_n = 1\}$ be subdivision with smaller amount of points (t_n is removed). It is enough to show that $[\gamma_D] = [\gamma_{D'}]$, since then we may proceed by induction. First we show that $[\gamma_D] = [\gamma_E]$, where $E\{0 = t_0 < t_1 < \ldots < t'_{n-1} < t_n = 1\}$ is a subdivision with the same points as D, except $t'_{n-1} = (t'_{n-2} + t_n)/2$. Define for every subdivision D as above the continuous mapping $f_D\colon I \to S^1$ as following.

Let $D_n = \{0, 1/n, \ldots, i/n, n-1/n, 1\}$ be a standard regular subdivision of I. For any subdivision of $n+1$ points $D = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$ we let a_D be a piecewise linear mapping $a_D\colon I \to I$ which maps i/n to t_i and is linear on the subintervals $[t_i, t_{i+1}]$. In other words

$$a_D(ti/n + (1-t)(i+1)/n) = tt_i + (1-t)t_{i+1}, i = 0, \ldots, n-1.$$

We also define $b_D \colon I \to S^1$ by

$$b_D(x) = e^{2\pi i a_D(x)}$$

and notice that $b_D(0) = b_D(1)$, so b_D induces a mapping $c_D \colon S^1 \to S^1$ so that

$$c(e^{2\pi t}) = b_D(t).$$

It is easy to see that a_D and a_E are homotopic rel $\{0, 1\}$, so c_D is homotopic to c_E. It follows that

$$[\gamma_D] = (c_D)_*([\gamma_{D_n}]) = (c_E)_*([\gamma_{D_n}]) = [\gamma_E].$$

Now for subdivision E we have that $\alpha_{n-2} \cdot \alpha_{n-1} = \alpha'_{n-2}$, where α'_{n-2} is the last summond in $\gamma_{D'}$. Exercise 4.7 implies that

$$[\gamma_D] = [\gamma_{D'}].$$

Since $[\gamma_{D_0}]$ is known to be the generator of $H_1(S^1)$, the last claim follows.

6. a) Suppose K is a simplicial complex and L_1 and L_2 are subcomplexes of K such that $K = L_1 \cup L_2$. Show that $(|K|; |L_1|, |L_2|)$ is a proper triad. (Hint: use the equivalence of simplicial and singular homologies).

   b)Show that $(S^n; B_+, B_-)$ is a proper triad using a). Write down the Mayer-Vietoris sequence of this triad and use it to prove that $H_n(S^n) \cong H_{n-1}(S^{n-1})$ for $n > 1$.

   c) Can you prove that $(S^n; B_+, B_-)$ is a proper triad using the properties of the singular homology, such as homotopy axiom and Mayer-Vietoris sequence for the open covering by 2 sets?

Solution: a) Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_n(L_1 \cap L_2) & \longrightarrow & C_n(L_1) \oplus C_n(L_2) & \longrightarrow & C_n(L_1) + C_n(L_2) = C_K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k \oplus k} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & C_n(|L_1 \cap L_2|) & \longrightarrow & C_n(|L_1|) \oplus C_n(|L_2|) & \longrightarrow & C_n(|L_1|) + C_n(|L_2|) & \longrightarrow & 0,
\end{array}$$

where vertical mappings k are canonical embeddings of simplicial chain group into singular chain group and horizontal mappings are defined as usual in Mayer-Viatoris sequence. Notice that $|L_1| \cap |L_2| = |L_1 \cap L_2|$. This diagram induces the commutative diagram in homology

$$\begin{array}{ccccccccc}
H_n(L_1 \cap L_2) & \longrightarrow & H_n(L_1) \oplus H_n(L_2) & \longrightarrow & H_n(K) & \longrightarrow & H_{n-1}(L_1 \cap L_2) & \longrightarrow & H_n \\
\downarrow{\scriptstyle k_* \cong} & & \downarrow{\scriptstyle k_* \oplus k_* \cong} & & \downarrow{\scriptstyle k_*} & & \downarrow{\scriptstyle k_* \cong} & & \\
H_n(|L_1 \cap L_2|) & \longrightarrow & H_n(|L_1|) \oplus H_n(|L_2|) & \longrightarrow & H_n(C|L_1| + C|L_2|) & \longrightarrow & H_{n-1}(|L_1 \cap L_2|) & \longrightarrow & H_{n-1}
\end{array}$$

Five-lemma implies that $k_*\colon H_n(K)\to H_n(C|L_1|+C|L_2|)$ is an isomorphism. On the other hand there is a commutative diagram

$$\begin{array}{ccc} H_n(K) & \xrightarrow{k_*\cong} & H_n(C|L_1|+C|L_2|) \\ & \searrow^{k_*\cong} & \downarrow{\scriptstyle i_*} \\ & & H_n(|K|), \end{array}$$

which shows that $i_*\colon H_n(C|L_1|+C|L_2|)\to H_n(|K|)$ is an isomorphism.

b) Since $(S^n;B_+,B_-)$ is homeomorphic to $(|K|;|L_1|,|L_2|)$ for a Δ-complex $|K|$, which consists of two n-simplices U and V glued by their boundary, where U generates L_1 and V generates L_2, the first claim follows from a).

Since $B_+\cap B_- = S^{n-1}$, the exact Mayer-Vietoris sequence of the triad $(S^n;B_+,B_-)$ looks like

$$\ldots \longrightarrow H_n(B_+)\oplus H_n(B_-) \longrightarrow H_n(S^n) \xrightarrow{\partial} H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(B_+)\oplus H_{n-1}(B_-) \longrightarrow \ldots.$$

If $n>1$, then $H_n(B_+)=H_n(B_-)=H_{n-1}(B_+)=H_{n-1}(B_-)=0$, since both B_+ and B_- are contractible. Hence from exactness it follows that $\partial\colon H_n(S^n)\to H_{n-1}(S^{n-1})$ is an isomorphism.

c) Let

$$U_+ = S^n\setminus\{-e_{n+1}\}$$

$$U_- = S^n\setminus\{e_{n+1}\}.$$

Then $\{U_+,U_-\}$ is an open covering of S^n, in particular (S^n,B_+,B_-) is a proper triad. Moreover inclusions $i\colon S^{n-1}\hookrightarrow U_-\cap U_+ = S$, $i\colon B_+\colon U_+$, $i\colon B_-\colon U_-$ are all homotopy equivalences, hence induce isomorphisms in homology.
Consider the commutative diagram

$$\begin{array}{ccccccccc} H_n(S^{n-1}) & \longrightarrow & H_n(B_+)\oplus H_n(B_-) & \longrightarrow & H_n(C(B_+)+C(B_-)) & \longrightarrow & H_{n-1}(S^{n-1}) & \longrightarrow & H_{n-1}(B_+)\oplus \\ \downarrow{\scriptstyle i_*\cong} & & \downarrow{\scriptstyle i_*\oplus i_*\cong} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*\cong} & & \downarrow{\scriptstyle k} \\ H_n(S) & \longrightarrow & H_n(U_+)\oplus H_n(U_-) & \longrightarrow & H_n(C(U_+)+C(U_-)) & \longrightarrow & H_{n-1}(S^{n-1}) & \longrightarrow & H_{n-1}(U_+)\oplus \end{array}$$

which is induced by the standard looking diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(S^{n-1}) & \longrightarrow & C_n(B_+)\oplus C_n(B_-) & \longrightarrow & C_n(B_+)+C_n(B_+) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & C_n(S) & \longrightarrow & C_n(U_+)\oplus C_n(U_-) & \longrightarrow & C_n(U_+)+C_n(U_-) & \longrightarrow & 0. \end{array}$$

By the Five-Lemma we see that $i_*\colon H_n(C(B_+)+C(B_-))\to H_n(C(U_+)+C(U_-))$ is an isomorphism for all $n\in\mathbb{N}$. On the other hand $j-*\colon H_n(C(U_+)+C(U_-))\to H_n(S^n)$ is an isomorphism for all $n\in\mathbb{N}$. Hence their composition, which is mapping $H_n(C(B_+)+C(B_-))\to H_n(S^n)$ induced by inclusion of chain complexes, is an isomorphism.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.