Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 8
07.11-11.11.2011

1. Let X be a non-empty set. Define $C_n(X)$ to be the free abelian group generated on the set X^{n+1} for $n \geq 0$ and $C_n(X) = 0$ for $n < 0$. Prove that the definition

$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_n)$$

defines a boundary operator that makes the collection $C(X) = \{C_n(X), \partial\}$ a chain complex. Prove that $C(X)$ has an augmentation $\varepsilon \colon C_0(X) \to \mathbb{Z}$ defined by $\varepsilon(x) = 1$ on generators.

For a fixed $x \in X$ and every $n \geq 0$ define homomorphism $x \colon C_n(X) \to C_{n+1}(X)$ by

$$x(x_0, \ldots, x_n) = (x, x_0, \ldots, x_n).$$

Prove that

$$(\partial_{n+1} x + x \partial_n)(y) = \begin{cases} y, & \text{if } n \neq 0, \\ y - \varepsilon(y)x, & \text{if } n = 0. \end{cases}$$

for all $y \in C(X)$. Deduce that the complex $\widetilde{C}(X)$ is acyclic.

2. Suppose C, D are chain complexes and $f_n, g_n \colon C_n \to D_n$ homomorphisms defined for every $n \in \mathbb{Z}$. Suppose for every $n \in \mathbb{N}$ there exists a homomorphism $H_n \colon C_n \to D_{n+1}$ with the property

$$\partial_{n+1} H_n + H_{n-1} \partial_n = f_n - g_n \text{ for all } n \in \mathbb{Z}.$$

Prove that $f - g = \{f_n - g_n | n \in \mathbb{Z}\}$ is a chain mapping.
Deduce that if g is a chain mapping, also f is. In other words **mapping that is homotopic to a chain mapping is a chain mapping itself**.

3. Define a homotopy $H_n \colon C_n(X) \to C_{n+1}X$ by

$$H_n(\sigma) = \sigma_\sharp(H_n(\Delta_n)),$$

where $H_n(\Delta_n)$ is the image of $\mathrm{id} \colon \Delta_n \to \Delta_n$ under $H_n \colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n) \subset C_n(\Delta_n)$. Prove (using the corresponding property of $H_n \colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n)$) that H is a chain homotopy between id and barycentric subdivision operator $S \colon C(X) \to C(X)$.

4. Let

$$B_+ = \{x \in S^n | x_{n+1} \geq 0\} \text{ and}$$

$$B_- = \{x \in S^n | x_{n+1} \leq 0\}.$$

Use homology and excision axioms to show that the inclusions $i\colon (B_+, S^{n-1}) \to (S^n, B_-)$ and $j\colon (B_-, S^{n-1}) \to (S^n, B_+)$ induce isomorphism in relative homology (for all dimensions).

5. a) Suppose $U \subset \mathbb{R}^n$ is open and $x \in U$. Prove that
$$j_*\colon H_n(U, U \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$
for all $n \in \mathbb{N}$. Here j is an obvious inclusion of pairs.
b) Suppose $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are both open and there is a homeomorphism $f\colon U \to V$. Prove that $n = m$.(Hint: remove a point)

6. Suppose $f\colon \overline{B}^n \to \overline{B}^n$ is a homeomorphism. Show that f maps interior B^n onto itself and the boundary S^{n-1} also onto itself. (Hint: remove a point).

7. Show that $U = S\backslash\{e_{n+1}\}$ is homeomorphic to $\mathbb{R}^n$ via stereographic projection through the north pole e_{n+1}.
Stereographic projection of the point $y \in U$ is defined to be the unique point in $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ which lies on the line spanned by y and e_{n+1}. Construct the explicit formula for the stereographic projection and its inverse.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.