Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 6
17.10-21.10.2011

1. Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes and chain mappings.

Consider the sequence

$$\dots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\partial} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\partial} H_{n-1}(C') \longrightarrow \dots,$$

where ∂ is a boundary homology operator, defined as usual.

a) Prove that

$$\operatorname{Ker} \partial \subset \operatorname{Im} g_*.$$

b) Prove the exactness of the sequence at $H_n(C')$.

2. Suppose (X, A, B) is a topological triple.
a)Prove that

$$0 \longrightarrow C(A,B) \xrightarrow{i_\sharp} C(X,B) \xrightarrow{j_\sharp} C(X,A) \longrightarrow 0,$$

where $i\colon (A,B) \to (X,B)$ and $j\colon (X,B) \to (X,A)$ are obvious inclusions, is a short exact sequence. Deduce the existence of the long exact sequence

$$\dots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial'} H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \longrightarrow \dots.$$

b) Show that for the boundary operators of the long exact homology sequences of the pair (X, A) and of the triple (X, A, B) there is a commutative diagram

$$\begin{array}{ccc} & & H_n(A) \\ & \overset{\partial}{\nearrow} & \\ H_{n+1}(X,A) & & \Big\downarrow i_* \\ & \underset{\partial'}{\searrow} & \\ & & H_n(A,B), \end{array}$$

where $i\colon A \to (A,B)$ is an inclusion. (Hint: use naturality of the long exact homology sequence.)

3. Prove the second part of the Five-Lemma: Suppose the diagram of groups and homomorphisms

$$\begin{array}{ccccccccc} G_1 & \xrightarrow{\alpha_1} & G_2 & \xrightarrow{\alpha_2} & G_3 & \xrightarrow{\alpha_3} & G_4 & \xrightarrow{\alpha_4} & G_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ H_1 & \xrightarrow{\beta_1} & H_2 & \xrightarrow{\beta_2} & H_3 & \xrightarrow{\beta_3} & H_4 & \xrightarrow{\beta_4} & H_5 \end{array}$$

   is commutative, rows are exact, f_5 is injective and f_2, f_4 are surjective. Then f_3 is surjective.

4. Suppose the chain complex C is a direct sum of complexes $(C_a)_{a\in\mathcal{A}}$. Prove that the inclusion mappings $i_a\colon C_a \to C$ induce a chain isomorphism

$$((i_a)_*)_{a\in\mathcal{A}}\colon \oplus_{a\in\mathcal{A}} H_n(C_a) \to H_n(C)$$

   for every $n \in \mathbb{N}$.

5. a) Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

   is a short exact sequence of chain complexes and chain mappings. Prove that if any two of complexes $C', C, \overline{C}$ are acyclic, also the third one is acyclic.

   b) Suppose

$$\begin{array}{ccccccccc} \dots & & 0 & & 0 & & 0 & & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_{n+1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & B_{n-1} & \longrightarrow & B_n & \longrightarrow & B_{n+1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & C_{n-1} & \longrightarrow & C_n & \longrightarrow & C_{n+1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & & 0 & & 0 & & 0 & & \dots \end{array}$$

   is a commutative diagram of abelian groups and homomorphisms. Assume that all columns are exact and the middle row (of B's) is exact. Prove that the upper row (A's) is exact if and only if lower row (C's) is exact. (Hint: all three horizontal sequences are chain complexes).

6. Suppose $f\colon X \to Y$ is continuous and X, Y are both path-connected and non-empty spaces. Show that $f_*\colon H_0(X) \to H_0(Y)$ is an isomorphism.

7. Suppose C is a chain complex with an augmentation ε. Prove that the sequence

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

is a chain complex C', and $H_n(\widetilde{C}) = H_n(C')$ for all $n \in \mathbb{Z}$. This gives another interpretation of reduced homology groups.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.